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INTRODUCTION TO CONFORMABLE ISO-DIFFERENTIAL CALCULUS

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Abstract

In this paper we introduce conformable iso-derivatives and conformable iso-integrals. They are deducted some of their properties. They are introduced conformable iso-exponential function, conformable iso-trigonometric functions and conformable iso-hyperbolic functions.

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1 Introduction

In the control theory, a proportional derivative controller for output u at time t with two tuning parameters has the algorithm

$$u(t) = k_p E(t) + k_d \frac{d}{dt} E(t),$$

where k_p is the proportional gain, k_d is the derivative gain, and E is the error between the state variable and the process variable.

The main aim of this paper is to be reflected the proportional derivative in the framewrok of iso-differential calculus. In the next Section we introduce conformable iso-derivative and we deduct some of its properties. In Section 3 we define conformable iso-exponential function and they are given some of its properties. Using the definition for conformable iso-exponential function, they are introduced conformable iso-hyperbolic functions and conformable iso-trigonometric functions in Section 4 and Section 5, respectively. In Section 6 are considered some applications to second order conformable iso-differential equations with constant coefficients. In Section 7 are defined higher order conformable iso-derivatives. Conformable iso-integrals and some of their properties are introduced in Section 8. In Section 9 are considered some applications to first order linear conformable iso-differential equations.

2 Definition of Conformable Iso-Derivative. Properties

Throughout of this paper, suppose that A is a subset of $\mathbb{R}$ and $\hat{T} : A \to \mathbb{R}$ is a continuous positive function which is differentiable on A, $t\hat{T}(t) \in A$, $\frac{t}{\hat{T}(t)} \in A$ and

$$\hat{T}(t) - t\hat{T}'(t) \neq 0$$

for any $t \in A$. Assume that $\hat{I}_1$ is the basic iso-unit of the set of iso-real numbers $\hat{\mathbb{R}}$. Let $\alpha \in [0,1]$ and let the functions $k_0, k_1 : [0,1] \times A \to [0,\infty)$ be continuous

such that

$$\begin{aligned} \lim_{\alpha\to 0+} k_1(\alpha,t) &= 1, \quad \lim_{\alpha\to 1-} k_1(\alpha,t) = 0, \quad \forall t\in A, \\ \lim_{\alpha\to 0+} k_0(\alpha,t) &= 0, \quad \lim_{\alpha\to 1-} k_0(\alpha,t) = 1, \quad \forall t\in A, \\ k_0(\alpha,t) &\neq 0, \quad \alpha\in(0,1], \quad \forall t\in A, \\ k_1(\alpha,t) &\neq 0, \quad \alpha\in[0,1), \quad \forall t\in A. \end{aligned} \tag{2.1}$$

Such functions k_0, k_1 exist. Really, the functions

$$k_0(\alpha,t) = \alpha\left(1+t^2\right)^{1-\alpha}, \quad k_1(\alpha,t) = (1-\alpha)\left(1+t^2\right)^{\alpha},$$

$\alpha\in[0,1]$, $t\in A$, satisfy (2.1).

Definition 2.1. *Let $\hat{f}: A\to\mathbb{R}$ is an iso-function of the first, second, third, fourth or fifth kind that is iso-differentiable at $t\in A$. Then the conformable iso-derivative of $\hat{f}$ of order α at t is defined by*

$$D^{\alpha\circledast}\hat{f}(t) = \hat{k}_1(\alpha,t)\hat{\times}\hat{f}(t) + \hat{k}_0(\alpha,t)\hat{\times}\hat{f}^{\circledast}(t),$$

where

$$\hat{k}_1(\alpha,t) = \frac{k_1(\alpha,t)}{\hat{T}(t)}, \quad \hat{k}_0(\alpha,t) = \frac{k_0(\alpha,t)}{\hat{T}(t)}, \quad t\in A.$$

Remark 2.2. *When $\alpha = 0$, we have*

$$D^{0\circledast}\hat{f}(t) = \hat{f}(t).$$

When $\alpha = 1$, we have

$$D^{1\circledast}\hat{f}(t) = \hat{f}^{\circledast}(t).$$

Below we suppose $\hat{f}: A\to\mathbb{R}$ is an iso-functions of the first, second, third, fourth or fifth kind that is iso-differentiable at $t\in A$, and $\alpha\in(0,1]$.

1. If $\hat{f}$ is an iso-function of the first kind, i.e.,

$$\hat{f}^{\wedge}(\hat{t}) = \frac{f(t)}{\hat{T}(t)},$$

then its conformable iso-derivative of order α at t is represented in the form

$$D^{\alpha\circledast}\hat{f}^{\wedge}(\hat{t}) = k_1(\alpha,t)\frac{f(t)}{\hat{T}(t)} + k_0(\alpha,t)\frac{f'(t)\hat{T}(t) - f(t)\hat{T}'(t)}{\hat{T}(t)\left(\hat{T}(t) - t\hat{T}'(t)\right)}.$$

2. If $\hat{f}$ is an iso-function of the second kind, i.e.,

$$\hat{f}^{\wedge}(t) = \frac{f\left(t\hat{T}(t)\right)}{\hat{T}(t)},$$

then its conformable iso-derivative of order α at t is represented in the form

$$\begin{aligned} D^{\alpha\circledast}\hat{f}^{\wedge}(t) &= k_1(\alpha,t)\frac{f\left(t\hat{T}(t)\right)}{\hat{T}(t)} \\ &\quad + k_0(\alpha,t)\frac{f'\left(t\hat{T}(t)\right)\left(\left(\hat{T}(t)\right)^2 + t\hat{T}(t)\hat{T}'(t)\right) - f\left(t\hat{T}(t)\right)\hat{T}'(t)}{\hat{T}(t)\left(\hat{T}(t) - t\hat{T}'(t)\right)}. \end{aligned}$$

3. If $\hat{f}$ is an iso-function of the third kind, i.e.,

$$\hat{f}(\hat{t}) = \frac{f\left(\frac{t}{\hat{T}(t)}\right)}{\hat{T}(t)},$$

then its conformable iso-derivative of order α at t is represented in the form

$$\begin{aligned} D^{\alpha\circledast}\hat{f}(\hat{t}) &= k_1(\alpha,t)\frac{f\left(\frac{t}{\hat{T}(t)}\right)}{\hat{T}(t)} \\ &\quad + k_0(\alpha,t)\frac{f'\left(\frac{t}{\hat{T}(t)}\right)\left(\hat{T}(t) - t\hat{T}'(t)\right) - f\left(\frac{t}{\hat{T}(t)}\right)\hat{T}(t)\hat{T}'(t)}{\left(\hat{T}(t)\right)^2\left(\hat{T}(t) - t\hat{T}'(t)\right)}. \end{aligned}$$

4. If $\hat{f}$ is an iso-function of the fourth kind, i.e.,

$$f^{\wedge}(t) = f\left(t\hat{T}(t)\right),$$

then its conformable iso-derivative of order α at t is given by

$$\begin{aligned} D^{\alpha\circledast}f^{\wedge}(t) &= k_1(\alpha,t)f\left(t\hat{T}(t)\right) \\ &\quad + k_0(\alpha,t)\frac{f'\left(t\hat{T}(t)\right)\hat{T}(t)\left(\hat{T}(t) + t\hat{T}'(t)\right)}{\hat{T}(t) - t\hat{T}'(t)}. \end{aligned}$$

5. If $\hat{f}$ is an iso-function of the fifth kind, i.e.,

$$f^{\vee}(t) = f\left(\frac{t}{\hat{T}(t)}\right),$$

then its conformable iso-derivative of order α at t is represented by

$$D^{\alpha\circledast} f^{\vee}(t) = k_1(\alpha,t) f\left(\frac{t}{\hat{T}(t)}\right) + \frac{k_0(\alpha,t)}{\hat{T}(t)} f'\left(\frac{t}{\hat{T}(t)}\right).$$

Example 2.3. *Let* $A = [0,\infty)$, $\hat{T}(t) = t+1$, $f(t) = t$, $t \in A$, *and*

$$k_1(\alpha,t) = (1-\alpha)t^{4\alpha}, \quad k_0(\alpha,t) = \alpha t^{4(1-\alpha)}, \quad \alpha \in (0,1], \quad t \in A.$$

Let $t \in A$ *be arbitrarily chosen. We will find*

$$D^{\frac{1}{4}\circledast}\hat{f}^{\wedge}(\hat{t}), \quad D^{\frac{1}{4}\circledast}\hat{f}^{\wedge}(t), \quad D^{\frac{1}{4}\circledast}\hat{f}(\hat{t}), \quad D^{\frac{1}{4}\circledast}f^{\wedge}(t), \quad D^{\frac{1}{4}\circledast}f^{\vee}(t).$$

Firstly we will determine the corresponding iso-functions of the first, second, third, fourth and fifth kind. We have

$$\begin{aligned}
k_1\left(\frac{1}{4},t\right) &= \frac{3}{4}t,\\
k_0\left(\frac{1}{4},t\right) &= \frac{1}{4}t,\\
\hat{f}^{\wedge}(\hat{t}) &= \frac{t}{t+1},\\
\hat{f}^{\wedge}(t) &= t,\\
\hat{f}(\hat{t}) &= \frac{t}{(t+1)^2},\\
f^{\wedge}(t) &= t^2+t,\\
f^{\vee}(t) &= \frac{t}{t+1}.
\end{aligned}$$

Then

$$\begin{aligned}
\hat{f}^{\wedge\circledast}(\hat{t}) &= \frac{1}{t+1}, \quad D^{\frac{1}{4}\circledast}\hat{f}^{\wedge}(\hat{t}) = \frac{3t^2+t}{4(t+1)}, \\
\hat{f}^{\wedge\circledast}(t) &= t+1, \quad D^{\frac{1}{4}\circledast}\hat{f}^{\wedge}(t) = t^2+\frac{1}{4}t, \\
\hat{f}^{\circledast}(\hat{t}) &= \frac{1-t}{(t+1)^2}, \quad D^{\frac{1}{4}\circledast}\hat{f}(\hat{t}) = \frac{2t^2+t}{4(t+1)}, \\
f^{\wedge\circledast}(t) &= 2t^2+3t+1, \quad D^{\frac{1}{4}\circledast}f^{\wedge}(t) = \frac{t(t+1)(5t+1)}{4}, \\
f^{\vee\circledast}(t) &= \frac{1}{t+1}, \quad D^{\frac{1}{4}\circledast}f^{\vee}(t) = \frac{3t^2+t}{4(t+1)}.
\end{aligned}$$

Below we will list some of the properties of the conformable iso-derivative of order α.

Theorem 2.4. *Let $\hat{f}, \hat{g} : A \to \mathbb{R}$ be iso-functions of the first, second, third, fourth or fifth kind which are iso-differentiable at $t \in A$. Then*

$$D^{\alpha\circledast}\left(\hat{a}\hat{\times}\hat{f}+\hat{b}\hat{\times}\hat{g}\right)(t) = \hat{a}\hat{\times}D^{\alpha\circledast}\hat{f}(t)+\hat{b}\hat{\times}D^{\alpha\circledast}\hat{g}(t)$$

for any $\hat{a} = \frac{a}{T}$, $\hat{b} = \frac{b}{T(t)}$, $a, b \in \mathbb{R}$.

Proof. We have

$$\begin{aligned}
D^{\alpha\circledast}\left(\hat{a}\hat{\times}\hat{f}+\hat{b}\hat{\times}\hat{g}\right)(t) &= \hat{k}_1(\alpha,t)\hat{\times}\left(\hat{a}\hat{\times}\hat{f}+\hat{b}\hat{\times}\hat{g}\right)(t) \\
&\quad +\hat{k}_0(\alpha,t)\hat{\times}\left(\hat{a}\hat{\times}\hat{f}+\hat{b}\hat{\times}\hat{g}\right)^{\circledast}(t) \\
&= \hat{k}_1(\alpha,t)\hat{\times}\left(\hat{a}\hat{\times}\hat{f}(t)+\hat{b}\hat{\times}\hat{g}(t)\right) \\
&\quad +\hat{k}_0(\alpha,t)\hat{\times}\left(\hat{a}\hat{\times}\hat{f}^{\circledast}(t)+\hat{b}\hat{\times}\hat{g}^{\circledast}(t)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \hat{a}\hat{\times}\left(\hat{k}_1(\alpha,t)\hat{\times}\hat{f}(t)+\hat{k}_0(\alpha,t)\hat{\times}\hat{f}^{\circledast}(t)\right)\\
&\quad+\hat{b}\hat{\times}\left(\hat{k}_1(\alpha,t)\hat{\times}\hat{g}(t)+\hat{k}_0(\alpha,t)\hat{\times}\hat{g}^{\circledast}(t)\right)\\
&= \hat{a}\hat{\times}D^{\alpha\circledast}\hat{f}(t)+\hat{b}\hat{\times}D^{\alpha\circledast}\hat{g}(t).
\end{aligned}
$$

This completes the proof. □

Theorem 2.5. *Let $\hat{f}$, $\hat{g} : A \to \mathbb{R}$ be iso-functions of the first, second, third, fourth or fifth kind which are iso-differentiable at t. Then*

$$
\begin{aligned}
D^{\alpha\circledast}\left(\hat{f}\hat{\times}\hat{g}\right)(t) &= \hat{f}(t)\hat{\times}D^{\alpha\circledast}\hat{g}(t)+\hat{g}(t)\hat{\times}D^{\alpha\circledast}\hat{f}(t)\\
&\quad+\hat{f}(t)\hat{\times}\hat{g}(t)\hat{\times}\left(\frac{1}{\left(\hat{T}(t)\right)^{3}}\hat{\times}\hat{T}^{\circledast}(t)-\hat{k}_1(\alpha,t)\right).
\end{aligned}
$$

Proof. Using the definition for conformable iso-derivative of order α and the rule for the iso-derivative of the iso-product of two iso-functions, we get

$$
\begin{aligned}
D^{\alpha\circledast}\left(\hat{f}\hat{\times}\hat{g}\right)(t) &= \hat{k}_1(\alpha,t)\hat{\times}\left(\hat{f}\hat{\times}\hat{g}\right)(t)\\
&\quad+\hat{k}_0(\alpha,t)\hat{\times}\left(\hat{f}\hat{\times}\hat{g}\right)^{\circledast}(t)\\
&= \hat{k}_1(\alpha,t)\hat{\times}\left(\hat{f}\hat{\times}\hat{g}\right)(t)\\
&\quad+\hat{k}_0(\alpha,t)\hat{\times}\Bigg(\hat{f}(t)\hat{\times}\hat{g}^{\circledast}(t)+\hat{g}(t)\hat{\times}\hat{f}^{\circledast}(t)\\
&\quad+\left(\frac{1}{\left(\hat{T}(t)\right)^{3}}\right)\hat{\times}\hat{f}(t)\hat{\times}\hat{g}(t)\hat{\times}\hat{T}^{\circledast}(t)\Bigg)\\
&= \hat{f}(t)\hat{\times}\left(\hat{k}_1(\alpha,t)\hat{\times}\hat{g}(t)+\hat{k}_0(\alpha,t)\hat{\times}\hat{g}^{\circledast}(t)\right)
\end{aligned}
$$

$$+\hat{g}(t)\hat{\times}\left(\hat{k}_1(\alpha,t)\hat{\times}\hat{f}(t)+\hat{k}_0(\alpha,t)\hat{\times}\hat{f}^{\circledast}(t)\right)$$

$$-\hat{k}_1(\alpha,t)\hat{\times}\hat{f}(t)\hat{\times}\hat{g}(t)$$

$$+\left(\frac{1}{\left(\hat{T}(t)\right)^{\hat{3}}}\right)\hat{\times}\hat{f}(t)\hat{\times}\hat{g}(t)\hat{\times}\hat{T}^{\circledast}(t)$$

$$= \hat{f}(t)\hat{\times}D^{\alpha\circledast}\hat{g}(t)+\hat{g}(t)\hat{\times}D^{\alpha\circledast}\hat{f}(t)$$

$$+\hat{f}(t)\hat{\times}\hat{g}(t)\hat{\times}\left(\left(\frac{1}{\left(\hat{T}(t)\right)^{\hat{3}}}\right)\hat{\times}\hat{T}^{\circledast}(t)-\hat{k}_1(\alpha,t)\right).$$

This completes the proof. □

Theorem 2.6. *Let $\hat{f}$, $\hat{g} : A \to \mathbb{R}$ be iso-functions of the first, second, third, fourth or fifth kind which are iso-differentiable at t and $\hat{g}(t) \neq 0$. Then*

$$D^{\alpha\circledast}\left(\hat{f}\nearrow\hat{g}\right)(t)=\left(\hat{g}(t)\hat{\times}\hat{f}^{\circledast}(t)-\hat{f}(t)\hat{\times}\hat{g}^{\circledast}(t)\right)\nearrow\left((\hat{g}(t))^{\hat{2}}\right)$$

$$+\left(\hat{f}(t)\hat{\times}\hat{g}(t)\hat{\times}\left(\hat{k}_1(\alpha,t)-\left(\frac{1}{\left(\hat{T}(t)\right)^{\hat{3}}}\right)\hat{\times}\hat{T}^{\circledast}(t)\right)\right)\nearrow\left((\hat{g}(t))^{\hat{2}}\right).$$

Proof. Using the definition for conformable iso-derivative of order α and the rule for the iso-derivative of the quotient of two iso-functions, we get

$$D^{\alpha\circledast}\left(\hat{f}\nearrow\hat{g}\right)(t)=\hat{k}_1(\alpha,t)\hat{\times}\left(\hat{f}\nearrow\hat{g}\right)(t)+\hat{k}_0(\alpha,t)\hat{\times}\left(\hat{f}\nearrow\hat{g}\right)^{\circledast}(t)$$

$$= \hat{k}_1(\alpha,t)\hat{\times}\left(\hat{f}\nearrow\hat{g}\right)(t)$$

$$+\hat{k}_0(\alpha,t)\hat{\times}\left(\left(\left(\hat{g}(t)\hat{\times}\hat{f}^{\circledast}(t)-\hat{f}(t)\hat{\times}\hat{g}^{\circledast}(t)\right)\nearrow\left((\hat{g}(t))^{\hat{2}}\right)\right)\right.$$

$$
\begin{aligned}
&- \left(\left(\frac{1}{\left(\hat{T}(t) \right)^{\hat{3}}} \right) \hat{\times} \hat{f}(t) \hat{\times} \hat{g}(t) \hat{\times} \hat{T}^{\circledast}(t) \right) \nearrow \left((\hat{g}(t))^{\hat{2}} \right) \Bigg) \\
= \; & \left(\hat{g}(t) \hat{\times} \left(\hat{k}_1(\alpha, t) \hat{\times} \hat{f}(t) + \hat{k}_0(\alpha, t) \hat{\times} \hat{f}^{\circledast}(t) \right) \right) \nearrow \left((\hat{g}(t))^{\hat{2}} \right) \\
& - \left(\hat{f}(t) \hat{\times} \left(\hat{k}_1(\alpha, t) \hat{\times} \hat{g}(t) + \hat{k}_0(\alpha, t) \hat{\times} \hat{g}^{\circledast}(t) \right) \right) \nearrow \left((\hat{g}(t))^{\hat{2}} \right) \\
& + \left(\hat{k}_1(\alpha, t) \hat{\times} \hat{f}(t) \hat{\times} \hat{g}(t) \right) \nearrow \left((\hat{g}(t))^{\hat{2}} \right) \\
& - \left(\left(\frac{1}{\left(\hat{T}(t) \right)^{\hat{3}}} \right) \hat{\times} \hat{f}(t) \hat{\times} \hat{g}(t) \hat{\times} \hat{T}^{\circledast}(t) \right) \nearrow \left((\hat{g}(t))^{\hat{2}} \right) \Bigg) \\
= \; & \left(\hat{g}(t) \hat{\times} \hat{f}^{\circledast}(t) - \hat{f}(t) \hat{\times} \hat{g}^{\circledast}(t) \right) \nearrow \left((\hat{g}(t))^{\hat{2}} \right) \\
& + \left(\hat{f}(t) \hat{\times} \hat{g}(t) \hat{\times} \left(\hat{k}_1(\alpha, t) - \left(\frac{1}{\left(\hat{T}(t) \right)^{\hat{3}}} \right) \hat{\times} \hat{T}^{\circledast}(t) \right) \right) \nearrow \left((\hat{g}(t))^{\hat{2}} \right).
\end{aligned}
$$

This completes the proof. □

3 The Conformable Iso-Exponential Function

Suppose that $\hat{f}$, $\hat{g}$, $\hat{h} : A \to \mathbb{R}$ are iso-functions of the first, second, third, fourth or fifth kind. Let $s, t, r \in A$.

Definition 3.1. *The conformable iso-function $E_{\hat{f}}(t, s)$ is defined as follows*

$$
E_{\hat{f}}(t, s) = e^{\hat{\int}_s^t ((\hat{T}(y)(\hat{f}(y) - \hat{k}_1(\alpha, y))) \nearrow (\hat{k}_0(\alpha, y))) \hat{\times} d\hat{y}}.
$$

Using the definition of the iso-integral, we find

$$\begin{aligned} E_{\hat{f}}(t,s) &= e^{\int_s^t \frac{\hat{T}(y)}{(\hat{T}(y))^2}\frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}(\hat{T}(y))^2 d\hat{y}} \\ &= e^{\int_s^t \hat{T}(y)\frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}d\left(\frac{y}{\hat{T}(y)}\right)} \\ &= e^{\int_s^t \hat{T}(y)\frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{(\hat{T}(y))^2}\right)dy} \\ &= e^{\int_s^t \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy}. \end{aligned}$$

Below we will list some of the properties of the conformable iso-exponential function.

Theorem 3.2. *We have*

$$D_t^{\alpha\circledast}E_{\hat{f}}(t,s) = \hat{f}(t)\hat{\times}E_{\hat{f}}(t,s).$$

Proof. Note that

$$\begin{aligned} \left(E_{\hat{f}}(t,s)\right)_t^{\circledast} &= \left(\hat{d}E_{\hat{f}}(t,s)\right) \nearrow \left(\hat{d}\hat{t}\right) \\ &= \frac{\hat{d}E_{\hat{f}}(t,s)}{\hat{T}(t)\hat{d}\hat{t}} \\ &= \frac{dE_{\hat{f}}(t,s)}{\hat{T}(t)d\hat{t}} \\ &= \frac{dE_{\hat{f}}(t,s)}{\hat{T}(t)\frac{\hat{T}(t)-t\hat{T}'(t)}{(\hat{T}(t))^2}dt} \\ &= \frac{\hat{T}(t)}{\hat{T}(t)-t\hat{T}'(t)}\frac{d}{dt}E_{\hat{f}}(t,s) \\ &= \frac{\hat{T}(t)}{\hat{T}(t)-t\hat{T}'(t)} \end{aligned}$$

$$\times \frac{d}{dt}\left(e^{\int_s^t \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy}\right)$$

$$= \frac{\hat{T}(t)}{\hat{T}(t)-t\hat{T}'(t)}\frac{\hat{f}(t)-\hat{k}_1(\alpha,t)}{\hat{k}_0(\alpha,t)}\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}$$
$$\times e^{\int_s^t \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy}$$

$$= \frac{\hat{f}(t)-\hat{k}_1(\alpha,t)}{\hat{k}_0(\alpha,t)}E_{\hat{f}}(t,s).$$

Hence, using the definition of the conformable iso-derivative of order α, we obtain

$$\begin{aligned} D_t^{\alpha\circledast}E_{\hat{f}}(t,s) &= \hat{k}_1(\alpha,t)\hat{\times}E_{\hat{f}}(t,s)+\hat{k}_0(\alpha,t)\hat{\times}\left(E_{\hat{f}}(t,s)\right)_t^{\circledast} \\ &= \hat{k}_1(\alpha,t)\hat{\times}E_{\hat{f}}(t,s) \\ &\quad +\hat{k}_0(\alpha,t)\hat{T}(t)\frac{\hat{f}(t)-\hat{k}_1(\alpha,t)}{\hat{k}_0(\alpha,t)}E_{\hat{f}}(t,s) \\ &= \hat{k}_1(\alpha,t)\hat{\times}E_{\hat{f}}(t,s)+\hat{f}(t)\hat{\times}E_{\hat{f}}(t,s) \\ &\quad -\hat{k}_1(\alpha,t)\hat{\times}E_{\hat{f}}(t,s) \\ &= \hat{f}(t)\hat{\times}E_{\hat{f}}(t,s). \end{aligned}$$

This completes the proof. □

Corollary 3.3. *The conformable iso-exponential function is a solution of the conformable iso-differential equation*

$$D^{\alpha\circledast}y = \hat{f}(t)\hat{\times}y.$$

Theorem 3.4. *We have*

$$E_{\hat{f}}(t,s)E_{\hat{f}}(s,r) = E_{\hat{f}}(t,r).$$

Proof. We have

$$\begin{aligned} E_{\hat{f}}(t,s)E_{\hat{f}}(s,r) &= e^{\int_s^t \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\ &\quad \times e^{\int_r^s \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\ &= e^{\int_r^t \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\ &= E_{\hat{f}}(t,r). \end{aligned}$$

This completes the proof. □

Theorem 3.5. *We have*

$$E_{\hat{f}}(s,t) = \frac{1}{E_{\hat{f}}(t,s)}.$$

Proof. We have

$$\begin{aligned} E_{\hat{f}}(s,t) &= e^{\int_t^s \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\ &= e^{-\int_s^t \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\ &= \frac{1}{e^{\int_s^t \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy}} \\ &= \frac{1}{E_{\hat{f}}(t,s)}. \end{aligned}$$

This completes the proof. □

Definition 3.6. *1. We define the "conformable circle plus" $\oplus$ as follows.*

$$\hat{f} \oplus \hat{g} = \hat{f} + \hat{g} - \hat{k}_1.$$

2. We define the "conformable circle minus" as follows.

$$\ominus \hat{f} = -\hat{f} + 2\hat{k}_1$$

and

$$\hat{f} \ominus \hat{g} = \hat{f} \oplus (\ominus \hat{g}) .$$

Note that

1.

$$\begin{aligned} \hat{f} \ominus \hat{f} &= \hat{f} \oplus \left(\ominus \hat{f}\right) \\ &= \hat{f} + \left(\ominus \hat{f}\right) - \hat{k}_1 \\ &= \hat{f} - \hat{f} + 2\hat{k}_1 - \hat{k}_1 \\ &= \hat{k}_1 . \end{aligned}$$

2.

$$\begin{aligned} \hat{f} \ominus \hat{g} &= \hat{f} \oplus (\ominus \hat{g}) \\ &= \hat{f} + (\ominus \hat{g}) - \hat{k}_1 \\ &= \hat{f} - \hat{g} + 2\hat{k}_1 - \hat{k}_1 \\ &= \hat{f} - \hat{g} + \hat{k}_1 . \end{aligned}$$

3. $\hat{f} \oplus \left(\hat{g} \oplus \hat{h}\right) = \left(\hat{f} \oplus \hat{g}\right) \oplus \hat{h}$. Really,

$$\begin{aligned} \hat{f} \oplus \left(\hat{g} \oplus \hat{h}\right) &= \hat{f} + \left(\hat{g} \oplus \hat{h}\right) - \hat{k}_1 \\ &= \hat{f} + \hat{g} + \hat{h} - 2\hat{k}_1 \\ &= \left(\hat{f} + \hat{g} - \hat{k}_1\right) + \hat{h} - \hat{k}_1 \\ &= \left(\hat{f} \oplus \hat{g}\right) + \hat{h} - \hat{k}_1 \\ &= \left(\hat{f} \oplus \hat{g}\right) \oplus \hat{h} . \end{aligned}$$

4. We have

$$\hat{f} \oplus \hat{g} = \hat{g} \oplus \hat{f}.$$

Really,

$$\begin{aligned} \hat{f} \oplus \hat{g} &= \hat{f} + \hat{g} - \hat{k}_1 \\ &= \hat{g} + \hat{f} - \hat{k}_1 \\ &= \hat{g} \oplus \hat{f}. \end{aligned}$$

Theorem 3.7. *We have*

$$E_{\hat{f}}(t,s)E_{\hat{g}}(t,s) = E_{\hat{f}\oplus\hat{g}}(t,s).$$

Proof. We have

$$\begin{aligned} E_{\hat{f}}(t,s)E_{\hat{g}}(t,s) &= e^{\int_s^t \frac{f(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\ &\quad \times e^{\int_s^t \frac{\hat{g}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\ &= e^{\int_s^t \frac{\hat{f}(y)+\hat{g}(y)-2\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\ &= e^{\int_s^t \frac{(\hat{f}\oplus\hat{g})(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\ &= E_{\hat{f}\oplus\hat{g}}(t,s). \end{aligned}$$

This completes the proof. □

Theorem 3.8. *We have*

$$\frac{E_{\hat{f}}(t,s)}{E_{\hat{g}}(t,s)} = E_{\hat{f}\ominus\hat{g}}(t,s).$$

Proof. We have

$$\begin{aligned}
\frac{E_{\hat{f}}(t,s)}{E_{\hat{g}}(t,s)} &= \frac{e^{\int_s^t \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy}}{e^{\int_s^t \frac{\hat{g}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy}} \\
&= e^{\int_s^t \frac{\hat{f}(y)-\hat{g}(y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\
&= e^{\int_s^t \frac{\hat{f}(y)-\hat{g}(y)+\hat{k}_1(\alpha,y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\
&= e^{\int_s^t \frac{(\hat{f}\ominus\hat{g})(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\
&= E_{\hat{f}\ominus\hat{g}}(t,s).
\end{aligned}$$

This completes the proof. □

Theorem 3.9. *We have*

$$E_{\hat{f}}(s,t) = E_{\ominus\hat{f}}(t,s).$$

Proof. We have

$$\begin{aligned}
E_{\hat{f}}(s,t) &= e^{\int_t^s \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\
&= e^{-\int_s^t \frac{\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\
&= e^{\int_s^t \frac{-\hat{f}(y)+\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\
&= e^{\int_s^t \frac{-\hat{f}(y)+2\hat{k}_1(\alpha,y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\
&= e^{\int_s^t \frac{\ominus\hat{f}(y)-\hat{k}_1(\alpha,y)}{\hat{k}_0(\alpha,y)}\left(\frac{\hat{T}(y)-y\hat{T}'(y)}{\hat{T}(y)}\right)dy} \\
&= E_{\ominus\hat{f}}(t,s).
\end{aligned}$$

This completes the proof. □

4 Conformable Iso-Hyperbolic Functions

Suppose that $\hat{f}, \hat{g} : A \to \mathbb{R}$ are iso-functions of the first, second, third, fourth or fifth kind and $s, t \in A$.

Definition 4.1. *Define the conformable hyperbolic functions $Cosh_{\hat{f}}$ and $Sinh_{\hat{f}}$ as follows*

$$\begin{aligned} Cosh_{\hat{f}}(t,s) &= \frac{E_{\hat{f}}(t,s) + E_{-\hat{f}}(t,s)}{2}, \\ Sinh_{\hat{f}}(t,s) &= \frac{E_{\hat{f}}(t,s) - E_{-\hat{f}}(t,s)}{2}. \end{aligned}$$

Below we will list some of the properties of the conformable hyperbolic functions $Cosh_{\hat{f}}$ and $Sinh_{\hat{f}}$.

Theorem 4.2. *We have*

$$\begin{aligned} D_t^{\alpha\circledast} Cosh_{\hat{f}}(t,s) &= \hat{f}(t)\hat{\times} Sinh_{\hat{f}}(t,s), \\ D_t^{\alpha\circledast} Sinh_{\hat{f}}(t,s) &= \hat{f}(t)\hat{\times} Cosh_{\hat{f}}(t,s), \\ \Big(Cosh_{\hat{f}}(t,s)\Big)^2 - \Big(Sinh_{\hat{f}}(t,s)\Big)^2 &= E_{\hat{k}_1}(t,s). \end{aligned}$$

Proof.

$$\begin{aligned} D_t^{\alpha\circledast} Cosh_{\hat{f}}(t,s) &= D_t^{\alpha\circledast} \frac{E_{\hat{f}}(t,s) + E_{-\hat{f}}(t,s)}{2} \\ &= \frac{D_t^{\alpha\circledast} E_{\hat{f}}(t,s) + D_t^{\alpha\circledast} E_{-\hat{f}}(t,s)}{2} \\ &= \frac{\hat{f}(t)\hat{\times} E_{\hat{f}}(t,s) - \hat{f}(t)\hat{\times} E_{-\hat{f}}(t,s)}{2} \\ &= \hat{f}(t)\hat{\times} \frac{E_{\hat{f}}(t,s) - E_{-\hat{f}}(t,s)}{2} \\ &= \hat{f}(t)\hat{\times} Sinh_{\hat{f}}(t,s), \end{aligned}$$

$$
\begin{aligned}
D_t^{\alpha\circledast} Sinh_{\hat{f}}(t,s) &= D_t^{\alpha\circledast} \frac{E_{\hat{f}}(t,s) - E_{-\hat{f}}(t,s)}{2} \\
&= \frac{D_t^{\alpha\circledast} E_{\hat{f}}(t,s) - D_t^{\alpha\circledast} E_{-\hat{f}}(t,s)}{2} \\
&= \frac{\hat{f}(t)\hat{\times} E_{\hat{f}}(t,s) + \hat{f}(t)\hat{\times} E_{-\hat{f}}(t,s)}{2} \\
&= \hat{f}(t)\hat{\times} \frac{E_{\hat{f}}(t,s) + E_{-\hat{f}}(t,s)}{2} \\
&= \hat{f}(t)\hat{\times} Cosh_{\hat{f}}(t,s).
\end{aligned}
$$

Next,

$$
\begin{aligned}
\Big(Cosh_{\hat{f}}(t,s)\Big)^2 - \Big(Sinh_{\hat{f}}(t,s)\Big)^2 &= \left(\frac{E_{\hat{f}}(t,s) + E_{-\hat{f}}(t,s)}{2}\right)^2 \\
&\quad - \left(\frac{E_{\hat{f}}(t,s) - E_{-\hat{f}}(t,s)}{2}\right)^2 \\
&= \frac{1}{4}\Bigg(\Big(E_{\hat{f}}(t,s)\Big)^2 + 2E_{\hat{f}}(t,s)E_{-\hat{f}}(t,s) + \Big(E_{-\hat{f}}(t,s)\Big)^2 \\
&\quad - \Big(E_{\hat{f}}(t,s)\Big)^2 + 2E_{\hat{f}}(t,s)E_{-\hat{f}}(t,s) - \Big(E_{-\hat{f}}(t,s)\Big)^2 \Bigg) \\
&= E_{\hat{f}}(t,s)E_{-\hat{f}}(t,s) \\
&= E_{\hat{f}\oplus(-\hat{f})}(t,s) \\
&= E_{-\hat{k}_1}(t,s).
\end{aligned}
$$

This completes the proof. □

Definition 4.3. *Define the conformable hyperbolic functions* $Ch_{\hat{f}\hat{g}}$ *and* $Sh_{\hat{f}\hat{g}}$ *as*

follows.

$$Ch_{\hat{f}\hat{g}}(t,s) = \frac{E_{\hat{f}+\hat{g}}(t,s)+E_{\hat{f}-\hat{g}}(t,s)}{2},$$

$$Sh_{\hat{f}\hat{g}}(t,s) = \frac{E_{\hat{f}+\hat{g}}(t,s)-E_{\hat{f}-\hat{g}}(t,s)}{2}.$$

Below we will list some of the properties of the conformable hyperbolic functions $Ch_{\hat{f}\hat{g}}$ and $Sh_{\hat{f}\hat{g}}$.

Theorem 4.4. *We have*

$$D_t^{\alpha\circledast}Ch_{\hat{f}\hat{g}}(t,s) = \hat{f}(t)\hat{\times}Ch_{\hat{f}\hat{g}}(t,s)+\hat{g}(t)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s),$$

$$D_t^{\alpha\circledast}Sh_{\hat{f}\hat{g}}(t,s) = \hat{g}(t)\hat{\times}Ch_{\hat{f}\hat{g}}(t,s)+\hat{f}(t)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s),$$

$$\left(Ch_{\hat{f}\hat{g}}(t,s)\right)^2-\left(Sh_{\hat{f}\hat{g}}(t,s)\right)^2 = E_{2\hat{f}-\hat{k}_1}(t,s).$$

Proof. We have

$$\begin{aligned}
D_t^{\alpha\circledast}Ch_{\hat{f}\hat{g}}(t,s) &= D_t^{\alpha\circledast}\left(\frac{E_{\hat{f}+\hat{g}}(t,s)+E_{\hat{f}-\hat{g}}(t,s)}{2}\right)\\
&= \frac{D_t^{\alpha\circledast}E_{\hat{f}+\hat{g}}(t,s)+D_t^{\alpha\circledast}E_{\hat{f}-\hat{g}}(t,s)}{2}\\
&= \frac{1}{2}\Bigg(\left(\hat{f}(t)+\hat{g}(t)\right)\hat{\times}E_{\hat{f}+\hat{g}}(t,s)\\
&\quad +\left(\hat{f}(t)-\hat{g}(t)\right)\hat{\times}E_{\hat{f}-\hat{g}}(t,s)\Bigg)\\
&= \frac{1}{2}\Bigg(\hat{f}(t)\hat{\times}\left(E_{\hat{f}+\hat{g}}(t,s)+E_{\hat{f}-\hat{g}}(t,s)\right)\\
&\quad +\hat{g}(t)\hat{\times}\left(E_{\hat{f}+\hat{g}}(t,s)-E_{\hat{f}-\hat{g}}(t,s)\right)\Bigg)
\end{aligned}$$

$$= \hat{f}(t)\hat{\times}Ch_{\hat{f}\hat{g}}(t,s) + \hat{g}(t)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s),$$

$$D_t^{\alpha\circledast}Sh_{\hat{f}\hat{g}}(t,s) = D_t^{\alpha\circledast}\left(\frac{E_{\hat{f}+\hat{g}}(t,s) - E_{\hat{f}-\hat{g}}(t,s)}{2}\right)$$

$$= \frac{D_t^{\alpha\circledast}E_{\hat{f}+\hat{g}}(t,s) - D_t^{\alpha\circledast}E_{\hat{f}-\hat{g}}(t,s)}{2}$$

$$= \frac{1}{2}\Bigg(\left(\hat{f}(t)+\hat{g}(t)\right)\hat{\times}E_{\hat{f}+\hat{g}}(t,s)$$

$$-\left(\hat{f}(t)-\hat{g}(t)\right)\hat{\times}E_{\hat{f}-\hat{g}}(t,s)\Bigg)$$

$$= \frac{1}{2}\Bigg(\hat{f}(t)\hat{\times}\left(E_{\hat{f}+\hat{g}}(t,s) - E_{\hat{f}-\hat{g}}(t,s)\right)$$

$$+\hat{g}(t)\hat{\times}\left(E_{\hat{f}+\hat{g}}(t,.s) + E_{\hat{f}-\hat{g}}(t,s)\right)\Bigg)$$

$$= \hat{f}(t)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s) + \hat{g}(t)\hat{\times}Ch_{\hat{f}\hat{g}}(t,s).$$

Next,

$$\left(Ch_{\hat{f}\hat{g}}(t,s)\right)^2 - \left(Sh_{\hat{f}\hat{g}}(t,s)\right)^2 = \left(\frac{E_{\hat{f}+\hat{g}}(t,s) + E_{\hat{f}-\hat{g}}(t,s)}{2}\right)^2$$

$$-\left(\frac{E_{\hat{f}+\hat{g}}(t,s) - E_{\hat{f}-\hat{g}}(t,s)}{2}\right)^2$$

$$= \frac{1}{4}\Bigg(\left(E_{\hat{f}+\hat{g}}(t,s)\right)^2 + 2E_{\hat{f}+\hat{g}}(t,s)E_{\hat{f}-\hat{g}}(t,s)$$

$$+\left(E_{\hat{f}-\hat{g}}(t,s)\right)^2 - \left(E_{\hat{f}+\hat{g}}(t,s)\right)^2$$

$$+2E_{\hat{f}+\hat{g}}(t,s)E_{\hat{f}-\hat{g}}(t,s)$$

$$-\left(E_{\hat{f}-\hat{g}}(t,s)\right)^2\Bigg)$$

$$= E_{\hat{f}+\hat{g}}(t,s)E_{\hat{f}-\hat{g}}(t,s)$$

$$= E_{(\hat{f}+\hat{g})\oplus(\hat{f}-\hat{g})}(t,s).$$

Since

$$\left(\hat{f}+\hat{g}\right)\oplus\left(\hat{f}-\hat{g}\right) = \hat{f}+\hat{g}+\hat{f}-\hat{g}-\hat{k}_1$$

$$= 2\hat{f}-\hat{k}_1,$$

we conclude that

$$\left(Ch_{\hat{f}\hat{g}}(t,s)\right)^2-\left(Sh_{\hat{f}\hat{g}}(t,s)\right)^2 = E_{2\hat{f}-\hat{k}_1}(t,s).$$

This completes the proof. □

5 Conformable Iso-Trigonometric Functions

Suppose that $\hat{f},\hat{g}:A\to\mathbb{R}$ are iso-functions of the first, second, third, fourth or fifth kind and $s,t\in A$.

Definition 5.1. *Define the conformable iso-trigonometric functions* $\cos_{\hat{f}}$ *and* $\sin_{\hat{f}}$ *as follows.*

$$\cos_{\hat{f}}(t,s) = \frac{E_{i\hat{f}}(t,s)+E_{-i\hat{f}}(t,s)}{2},$$

$$\sin_{\hat{f}}(t,s) = \frac{E_{i\hat{f}}(t,s)-E_{-i\hat{f}}(t,s)}{2i}.$$

Below we will list some of the properties of the conformable iso-trigonometric functions $\cos_{\hat{f}}$ and $\sin_{\hat{f}}$.

Theorem 5.2. *We have*

$$\begin{aligned} D_t^{\alpha\circledast}\cos_{\hat{f}}(t,s) &= -\hat{f}(t)\hat{\times}\sin_{\hat{f}}(t,s), \\ D_t^{\alpha\circledast}\sin_{\hat{f}}(t,s) &= \hat{f}(t)\hat{\times}\cos_{\hat{f}}(t,s), \\ \left(\cos_{\hat{f}}(t,s)\right)^2+\left(\sin_{\hat{f}}(t,s)\right)^2 &= E_{-\hat{k}_1}(t,s). \end{aligned}$$

Proof. We have

$$\begin{aligned} D_t^{\alpha\circledast}\cos_{\hat{f}}(t,s) &= D_t^{\alpha\circledast}\left(\frac{E_{i\hat{f}}(t,s)+E_{-i\hat{f}}(t,s)}{2}\right) \\ &= \frac{D_t^{\alpha\circledast}E_{i\hat{f}}(t,s)+D_t^{\alpha\circledast}E_{-i\hat{f}}(t,s)}{2} \\ &= \frac{i\hat{f}(t)\hat{\times}E_{i\hat{f}}(t,s)-i\hat{f}(t)\hat{\times}E_{-i\hat{f}}(t,s)}{2} \\ &= \hat{f}(t)\hat{\times}\frac{E_{-i\hat{f}}(t,s)-E_{i\hat{f}}(t,s)}{2i} \\ &= -\hat{f}(t)\hat{\times}\sin_{\hat{f}}(t,s), \end{aligned}$$

$$\begin{aligned} D_t^{\alpha\circledast}\sin_{\hat{f}}(t,s) &= D_t^{\alpha\circledast}\left(\frac{E_{i\hat{f}}(t,s)-E_{-i\hat{f}}(t,s)}{2i}\right) \\ &= \frac{D_t^{\alpha\circledast}E_{i\hat{f}}(t,s)-D_t^{\alpha\circledast}E_{-i\hat{f}}(t,s)}{2i} \\ &= \frac{i\hat{f}(t)\hat{\times}E_{i\hat{f}}(t,s)+i\hat{f}(t)\hat{\times}E_{-i\hat{f}}(t,s)}{2i} \\ &= \hat{f}(t)\hat{\times}\frac{E_{i\hat{f}}(t,s)+E_{-i\hat{f}}(t,s)}{2} \\ &= \hat{f}(t)\hat{\times}\cos_{\hat{f}}(t,s). \end{aligned}$$

Next,

$$\begin{aligned}
\left(\cos_{\hat{f}}(t,s)\right)^2 + \left(\sin_{\hat{f}}(t,s)\right)^2 &= \left(\frac{E_{i\hat{f}}(t,s) + E_{-i\hat{f}}(t,s)}{2}\right)^2 \\
&\quad + \left(\frac{E_{i\hat{f}}(t,s) - E_{-i\hat{f}}(t,s)}{2i}\right)^2 \\
&= \frac{1}{4}\Bigg(\left(E_{i\hat{f}}(t,s)\right)^2 + 2E_{i\hat{f}}(t,s)E_{-i\hat{f}}(t,s) \\
&\quad + \left(E_{-i\hat{f}}(t,s)\right)^2 - \left(E_{i\hat{f}}(t,s)\right)^2 \\
&\quad + 2E_{i\hat{f}}(t,s)E_{-i\hat{f}}(t,s) \\
&\quad - \left(E_{-i\hat{f}}(t,s)\right)^2 \\
&= E_{i\hat{f}}(t,s)E_{-i\hat{f}}(t,s) \\
&= E_{(i\hat{f})\oplus(-i\hat{f})}(t,s).
\end{aligned}$$

Observe that

$$\begin{aligned}
(i\hat{f}) \oplus (-i\hat{f}) &= i\hat{f} - i\hat{f} - \hat{k}_1 \\
&= -\hat{k}_1.
\end{aligned}$$

Therefore

$$\left(\cos_{\hat{f}}(t,s)\right)^2 + \left(\sin_{\hat{f}}(t,s)\right)^2 = E_{-\hat{k}_1}(t,s).$$

This completes the proof. □

Definition 5.3. *Define the conformable iso-trigonometric functions* $C_{\hat{f}\hat{g}}$ *and* $S_{\hat{f}\hat{g}}$

as follows.

$$C_{\hat{f}\hat{g}}(t,s) = \frac{E_{\hat{f}+i\hat{g}}(t,s)+E_{\hat{f}-i\hat{g}}(t,s)}{2},$$

$$S_{\hat{f}\hat{g}}(t,s) = \frac{E_{\hat{f}+i\hat{g}}(t,s)-E_{\hat{f}-i\hat{g}}(t,s)}{2i}.$$

Below we will list some of the properties of the conformable iso-trigonometric functions $C_{\hat{f}\hat{g}}$ and $S_{\hat{f}\hat{g}}$.

Theorem 5.4. *We have*

$$D_t^{\alpha\circledast}C_{\hat{f}\hat{g}}(t,s) = \hat{f}(t)\hat{\times}C_{\hat{f}\hat{g}}(t,s)-\hat{g}(t)\hat{\times}S_{\hat{f}\hat{g}}(t,s),$$

$$D_t^{\alpha\circledast}S_{\hat{f}\hat{g}}(t,s) = \hat{f}(t)\hat{\times}S_{\hat{f}\hat{g}}(t,s)+\hat{g}(t)\hat{\times}C_{\hat{f}\hat{g}}(t,s),$$

$$\left(C_{\hat{f}\hat{g}}(t,s)\right)^2+\left(S_{\hat{f}\hat{g}}(t,s)\right)^2 = E_{2\hat{f}-\hat{k}_1}(t,s).$$

Proof. We have

$$\begin{aligned}
D_t^{\alpha\circledast}C_{\hat{f}\hat{g}}(t,s) &= D_t^{\alpha\circledast}\left(\frac{E_{\hat{f}+i\hat{g}}(t,s)+E_{\hat{f}-i\hat{g}}(t,s)}{2}\right)\\
&= \frac{D_t^{\alpha\circledast}E_{\hat{f}+i\hat{g}}(t,s)+D_t^{\alpha\circledast}E_{\hat{f}-i\hat{g}}(t,s)}{2}\\
&= \frac{\left(\hat{f}(t)+i\hat{g}(t)\right)\hat{\times}E_{\hat{f}+i\hat{g}}(t,s)+\left(\hat{f}(t)-i\hat{g}(t)\right)\hat{\times}E_{\hat{f}-i\hat{g}}(t,s)}{2}\\
&= \hat{f}(t)\hat{\times}\frac{E_{\hat{f}+i\hat{g}}(t,s)+E_{\hat{f}-i\hat{g}}(t,s)}{2}\\
&\quad +i\hat{g}(t)\hat{\times}\frac{E_{\hat{f}+i\hat{g}}(t,s)-E_{\hat{f}-i\hat{g}}(t,s)}{2}\\
&= \hat{f}(t)\hat{\times}C_{\hat{f}\hat{g}}(t,s)-\hat{g}(t)\hat{\times}S_{\hat{f}\hat{g}}(t,s),
\end{aligned}$$

$$
\begin{aligned}
D_t^{\alpha\circledast} S_{\hat{f}\hat{g}}(t,s) &= D_t^{\alpha\circledast}\left(\frac{E_{\hat{f}+i\hat{g}}(t,s) - E_{\hat{f}-i\hat{g}}(t,s)}{2i}\right) \\
&= \frac{D_t^{\alpha\circledast} E_{\hat{f}+i\hat{g}}(t,s) - D_t^{\alpha\circledast} E_{\hat{f}-i\hat{g}}(t,s)}{2i} \\
&= \frac{\left(\hat{f}(t)+i\hat{g}(t)\right)\hat{\times} E_{\hat{f}+i\hat{g}}(t,s) - \left(\hat{f}(t)-i\hat{g}(t)\right)\hat{\times} E_{\hat{f}-i\hat{g}}(t,s)}{2i} \\
&= \hat{f}(t)\hat{\times}\frac{E_{\hat{f}+i\hat{g}}(t,s) - E_{\hat{f}-i\hat{g}}(t,s)}{2i} \\
&\quad +\hat{g}(t)\hat{\times}\frac{E_{\hat{f}+i\hat{g}}(t,s) + E_{\hat{f}-i\hat{g}}(t,s)}{2} \\
&= \hat{f}(t)\hat{\times} S_{\hat{f}\hat{g}}(t,s) + \hat{g}(t)\hat{\times} C_{\hat{f}\hat{g}}(t,s).
\end{aligned}
$$

Next,

$$
\begin{aligned}
\left(C_{\hat{f}\hat{g}}(t,s)\right)^2 + \left(S_{\hat{f}\hat{g}}(t,s)\right)^2 &= \left(\frac{E_{\hat{f}+i\hat{g}}(t,s) + E_{\hat{f}-i\hat{g}}(t,s)}{2}\right)^2 \\
&\quad + \left(\frac{E_{\hat{f}+i\hat{g}}(t,s) - E_{\hat{f}-i\hat{g}}(t,s)}{2i}\right)^2 \\
&= \frac{1}{4}\Bigg(\left(E_{\hat{f}+i\hat{g}}(t,s)\right)^2 + 2E_{\hat{f}+i\hat{g}}(t,s)E_{\hat{f}-i\hat{g}}(t,s) \\
&\quad + \left(E_{\hat{f}-i\hat{g}}(t,s)\right)^2 - \left(E_{\hat{f}+i\hat{g}}(t,s)\right)^2 \\
&\quad +2E_{\hat{f}+i\hat{g}}(t,s)E_{\hat{f}-i\hat{g}}(t,s) \\
&\quad - \left(E_{\hat{f}-i\hat{g}}(t,s)\right)^2 \Bigg)
\end{aligned}
$$

$$= E_{\hat{f}+i\hat{g}}(t,s)E_{\hat{f}-i\hat{g}}(t,s)$$

$$= E_{(\hat{f}+i\hat{g})\oplus(\hat{f}-i\hat{g})}(t,s).$$

Observe that

$$\left(\hat{f}+i\hat{g}\right)\oplus\left(\hat{f}-i\hat{g}\right) = \hat{f}+i\hat{g}+\hat{f}-i\hat{g}-\hat{k}_1$$

$$= 2\hat{f}-\hat{k}_1.$$

Therefore

$$\left(C_{\hat{f}\hat{g}}(t,s)\right)^2+\left(S_{\hat{f}\hat{g}}(t,s)\right)^2 = E_{2\hat{f}-\hat{k}_1}(t,s).$$

This completes the proof. □

6 Higher Order Conformable Iso-Derivatives

Suppose that $n \in \mathbb{N}$, $\hat{f} : A \to \mathbb{R}$ is an iso-function of the first, second, third, fourth or fifth kind that is n times iso-differentiable at $t \in A$. Assume that k_0, k_1 satisfy (2.1) and $\hat{k}_0$, $\hat{k}_1$ are $n-1$ times iso-differentiable at $t \in A$.

Definition 6.1. *Define*

$$\left(D^{\alpha\circledast}\right)^n \hat{f}(t) = D^{\alpha\circledast}\left(D^{\alpha\circledast}\left(\ldots\left(D^{\alpha\circledast}\right)\right)\right)\hat{f}(t).$$

Remark 6.2. *If $n\alpha \in (0,1]$, then in the general case we have*

$$\left(D^{\alpha\circledast}\right)^n \hat{f}(t) \neq \left(D^{(n\alpha)\circledast}\right)\hat{f}(t).$$

We will see this in the following example. Let $A = (0,\infty)$, $\hat{T}(t) = t+1$, $t \in A$, $f(t) = 1$, $t \in A$, and

$$k_1(\alpha,t) = (1-\alpha)t^{4\alpha}, \quad k_0(\alpha,t) = \alpha t^{4(1-\alpha)}.$$

Then

$$\begin{aligned}
\hat{f}^{\wedge}(\hat{t}) &= \frac{1}{t+1}, \\
D^{\frac{1}{2}\circledast}\hat{f}^{\wedge}(\hat{t}) &= 0, \\
D^{\frac{1}{4}\circledast}\hat{f}^{\wedge}(\hat{t}) &= \frac{3t-t^3}{4(t+1)}, \\
D^{\frac{1}{4}\circledast}\left(D^{\frac{1}{4}\circledast}\right)\hat{f}^{\wedge}(\hat{t}) &= \frac{9t^2+3t^3-3t^4-3t^5-2t^6}{16(t+1)}, \quad t \in A.
\end{aligned}$$

7 Applications to Second Order Conformable Iso-Differential Equations with Constant Coefficients

Let $t, s \in A$.

1. Consider the equation

$$\left(D^{\alpha\circledast}\right)^2 y - \hat{a}^{\hat{2}}\hat{\times}y = 0,$$

where $\hat{a} = \frac{a}{T(t)}$, $a \in \mathbb{R}$. We will show that

$$y_1(t) = Cosh_{\hat{a}}(t,s) \quad \text{and} \quad y_2(t) = Sinh_{\hat{a}}(t,s)$$

are its solution. Really, we have

$$\begin{aligned}
D_t^{\alpha\circledast}y_1(t) &= \hat{a}\hat{\times}Sinh_{\hat{a}}(t,s), \\
\left(D_t^{\alpha\circledast}\right)^2 y_1(t) &= D_t^{\alpha\circledast}\left(\hat{a}\hat{\times}Sinh_{\hat{a}}(t,s)\right) \\
&= \hat{a}\hat{\times}D^{\alpha\circledast}\left(Sinh_{\hat{a}}(t,s)\right) \\
&= \hat{a}^{\hat{2}}\hat{\times}Cosh_{\hat{a}}(t,s) \\
&= \hat{a}^{\hat{2}}\hat{\times}y_1(t).
\end{aligned}$$

Then

$$\left(D_t^{\alpha\circledast}\right)^2 y_1(t) - \hat{a}^{\hat{2}} \hat{\times} y_1(t) = \hat{a}^{\hat{2}} \hat{\times} y_1(t) - \hat{a}^{\hat{2}} \hat{\times} y_1(t)$$

$$= 0.$$

Next,

$$D_t^{\alpha\circledast} y_2(t) = \hat{a} \hat{\times} Cosh_{\hat{a}}(t,s),$$

$$\left(D_t^{\alpha\circledast}\right)^2 y_2(t) = D_t^{\alpha\circledast}\left(\hat{a} \hat{\times} Cosh_{\hat{a}}(t,s)\right)$$

$$= \hat{a} \hat{\times} D_t^{\alpha\circledast}\left(Cosh_{\hat{a}}(t,s)\right)$$

$$= \hat{a}^{\hat{2}} \hat{\times} Sinh_{\hat{a}}(t,s)$$

$$= \hat{a}^{\hat{2}} \hat{\times} y_2(t).$$

Then

$$\left(D_t^{\alpha\circledast}\right)^2 y_2(t) - \hat{a}^{\hat{2}} \hat{\times} y_2(t) = \hat{a}^{\hat{2}} \hat{\times} y_2(t) - \hat{a}^{\hat{2}} \hat{\times} y_2(t)$$

$$= 0.$$

2. Consider the equation

$$\left(D^{\alpha\circledast}\right)^2 y + \hat{2} \hat{\times} \hat{b} \hat{\times} D^{\alpha\circledast} y + \hat{c} \hat{\times} y = 0,$$

where $\hat{2} = \frac{2}{\hat{T}(t)}$, $\hat{b} = \frac{b}{\hat{T}(t)}$, $\hat{c} = \frac{c}{\hat{T}(t)}$, $a, b, c \in \mathbb{R}$. We will show that

$$y_1(t) = Ch_{\hat{f}\hat{g}}(t,s), \quad y_2(t) = Sh_{\hat{f}\hat{g}}(t,s),$$

$$y_3(t) = C_{\hat{f}\hat{g}_1}(t,s) \quad y_4(t) = S_{\hat{f}\hat{g}_1}(t,s)$$

are its solutions, where

$$\hat{f} = -\hat{b}, \quad \hat{g}_1^{\hat{2}} = \hat{b}^{\hat{2}} - \hat{c}, \quad \hat{g}_1^{\hat{2}} = -\hat{b}^{\hat{2}} + \hat{c}.$$

Really, we have

$$
\begin{aligned}
D_t^{\alpha\circledast} y_1(t) &= \hat{f}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s) - \hat{g}\hat{\times} Sh_{\hat{f}\hat{g}}(t,s), \\
\left(D_t^{\alpha\circledast}\right)^2 y_2(t) &= D_t^{\alpha\circledast}\left(\hat{f}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s) + \hat{g}\hat{\times} Sh_{\hat{f}\hat{g}}(t,s)\right) \\
&= D_t^{\alpha\circledast}\left(\hat{f}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s)\right) \\
&\quad + D_t^{\alpha\circledast}\left(\hat{g}\hat{\times} Sh_{\hat{f}\hat{g}}(t,s)\right) \\
&= \hat{f}\hat{\times} D_t^{\alpha\circledast} Ch_{\hat{f}\hat{g}}(t,s) \\
&\quad + \hat{g}\hat{\times} D_t^{\alpha\circledast} Sh_{\hat{f}\hat{g}}(t,s) \\
&= \hat{f}\hat{\times}\left(\hat{f}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s) + \hat{g}\hat{\times} Sh_{\hat{f}\hat{g}}(t,s)\right) \\
&\quad + \hat{g}\hat{\times}\left(\hat{g}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s) + \hat{f}\hat{\times} Sh_{\hat{f}\hat{g}}(t,s)\right) \\
&= \hat{f}^{\hat{2}}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s) - \hat{2}\hat{\times}\hat{f}\hat{\times}\hat{g}\hat{\times} Sh_{\hat{f}\hat{g}}(t,s) \\
&\quad + \hat{g}^{\hat{2}}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left(D_t^{\alpha\circledast}\right)^2 y_1(t) + \hat{2}\hat{\times}\hat{b}\hat{\times} D_t^{\alpha\circledast} y_1(t) + \hat{c}\hat{\times} y_1(t) \\
&= \hat{f}^{\hat{2}}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s) + \hat{2}\hat{\times}\hat{f}\hat{\times}\hat{g}\hat{\times} Sh_{\hat{f}\hat{g}}(t,s) \\
&\quad + \hat{g}^{\hat{2}}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s) + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{f}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s) \\
&\quad + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}\hat{\times} Sh_{\hat{f}\hat{g}}(t,s) + \hat{c}\hat{\times} Ch_{\hat{f}\hat{g}}(t,s)
\end{aligned}
$$

$$= \left(\hat{f}^{\hat{2}} + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{f} + \hat{g}^{\hat{2}} + \hat{c}\right)\hat{\times}Ch_{\hat{f}\hat{g}}(t,s)$$

$$+ \left(\hat{2}\hat{\times}\hat{f}\hat{\times}\hat{g} + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}\right)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s)$$

$$= \left(\hat{b}^{\hat{2}} - \hat{2}\hat{\times}\hat{b}^{\hat{2}} + \hat{b}^{\hat{2}} - \hat{c} + \hat{c}\right)\hat{\times}Ch_{\hat{f}\hat{g}}(t,s)$$

$$+ \left(\hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g} - \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}\right)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s)$$

$$= 0.$$

Next,

$$D_t^{\alpha\circledast}y_2(t) = D_t^{\alpha\circledast}Sh_{\hat{f}\hat{g}}(t,s)$$

$$= \hat{g}\hat{\times}Ch_{\hat{f}\hat{g}}(t,s) + \hat{f}\hat{\times}Sh_{\hat{f}\hat{g}}(t,s),$$

$$\left(D_t^{\alpha\circledast}\right)^2 y_2(t) = D_t^{\alpha\circledast}\left(\hat{g}\hat{\times}Ch_{\hat{f}\hat{g}}(t,s) + \hat{f}\hat{\times}Sh_{\hat{f}\hat{g}}(t,s)\right)$$

$$= D_t^{\alpha\circledast}\left(\hat{g}\hat{\times}Ch_{\hat{f}\hat{g}}(t,s)\right)$$

$$+ D_t^{\alpha\circledast}\left(\hat{f}\hat{\times}Sh_{\hat{f}\hat{g}}(t,s)\right)$$

$$= \hat{g}\hat{\times}D_t^{\alpha\circledast}\left(Ch_{\hat{f}\hat{g}}(t,s)\right)$$

$$+ \hat{f}\hat{\times}D_t^{\alpha\circledast}\left(Sh_{\hat{f}\hat{g}}(t,s)\right)$$

$$= \hat{g}\hat{\times}\left(\hat{f}\hat{\times}Ch_{\hat{f}\hat{g}}(t,s) + \hat{g}\hat{\times}Sh_{\hat{f}\hat{g}}(t,s)\right)$$

$$+ \hat{f}\hat{\times}\left(\hat{g}\hat{\times}Ch_{\hat{f}\hat{g}}(t,s) + \hat{f}\hat{\times}Sh_{\hat{f}\hat{g}}(t,s)\right)$$

$$= \left(\hat{g}^{\hat{2}} + \hat{f}^{\hat{2}}\right)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s)$$

$$+ \hat{2}\hat{\times}\hat{g}\hat{\times}\hat{f}\hat{\times}Ch_{\hat{f}\hat{g}}(t,s).$$

Hence,

$$\begin{aligned}
&\left(D_t^{\alpha\circledast}\right)^2 y_2(t) + \hat{2}\hat{\times}\hat{b}\hat{\times}D_t^{\alpha\circledast} y_2(t) + \hat{c}\hat{\times}y_2(t) \\
&= \left(\hat{g}^{\hat{2}} + \hat{f}^{\hat{2}}\right)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s) + \hat{2}\hat{\times}\hat{g}\hat{\times}\hat{f}\hat{\times}Ch_{\hat{f}\hat{g}}(t,s) \\
&\quad +\hat{2}\hat{\times}\hat{b}\hat{\times}\left(\hat{g}\hat{\times}Ch_{\hat{f}\hat{g}}(t,s) + \hat{f}\hat{\times}Sh_{\hat{f}\hat{g}}(t,s)\right) \\
&\quad +\hat{c}\hat{\times}Sh_{\hat{f}\hat{g}}(t,s) \\
&= \left(\hat{g}^{\hat{2}} + \hat{f}^{\hat{2}} + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{f} + \hat{c}\right)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s) \\
&\quad + \left(\hat{2}\hat{\times}\hat{g}\hat{\times}\hat{f} + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}\right)\hat{\times}Ch_{\hat{f}\hat{g}}(t,s) \\
&= \left(\hat{b}^{\hat{2}} - \hat{c} + \hat{b}^{\hat{2}} - \hat{2}\hat{\times}\hat{b}^{\hat{2}} + \hat{c}\right)\hat{\times}Sh_{\hat{f}\hat{g}}(t,s) \\
&\quad + \left(-\hat{2}\hat{\times}\hat{g}\hat{\times}\hat{b} + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}\right)\hat{\times}Ch_{\hat{f}\hat{g}}(t,s) \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
D_t^{\alpha\circledast} y_3(t) &= \hat{f}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s) - \hat{g}_1\hat{\times}S_{\hat{f}\hat{g}_1}(t,s), \\
\left(D_t^{\alpha\circledast}\right)^2 y_3(t) &= D_t^{\alpha\circledast}\left(\hat{f}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s) - \hat{g}_1\hat{\times}S_{\hat{f}\hat{g}_1}(t,s)\right) \\
&= D_t^{\alpha\circledast}\left(\hat{f}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s)\right) \\
&\quad -D_t^{\alpha\circledast}\left(\hat{g}_1\hat{\times}S_{\hat{f}\hat{g}_1}(t,s)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \hat{f}\hat{\times}D_t^{\alpha\circledast}C_{\hat{f}\hat{g}_1}(t,s) \\
&\quad -\hat{g}_1\hat{\times}D_t^{\alpha\circledast}S_{\hat{f}\hat{g}_1}(t,s) \\
&= \hat{f}\hat{\times}\left(\hat{f}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s)-\hat{g}_1\hat{\times}S_{\hat{f}\hat{g}_1}(t,s)\right) \\
&\quad -\hat{g}_1\hat{\times}\left(\hat{g}_1\hat{\times}C_{\hat{f}\hat{g}_1}(t,s)+\hat{f}\hat{\times}S_{\hat{f}\hat{g}_1}(t,s)\right) \\
&= \hat{f}^{\hat{2}}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s)-\hat{2}\hat{\times}\hat{f}\hat{\times}\hat{g}_1\hat{\times}S_{\hat{f}\hat{g}_1}(t,s) \\
&\quad -\hat{g}_1^{\hat{2}}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left(D_t^{\alpha\circledast}\right)^2 y_3(t)+\hat{2}\hat{\times}\hat{b}\hat{\times}D_t^{\alpha\circledast}y_3(t)+\hat{c}\hat{\times}y_3(t) \\
&= \hat{f}^{\hat{2}}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s)-\hat{2}\hat{\times}\hat{f}\hat{\times}\hat{g}_1\hat{\times}S_{\hat{f}\hat{g}_1}(t,s) \\
&\quad -\hat{g}_1^{\hat{2}}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s)+\hat{2}\hat{\times}\hat{b}\hat{\times}\hat{f}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s) \\
&\quad -\hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}_1\hat{\times}S_{\hat{f}\hat{g}_1}(t,s)+\hat{c}\hat{\times}C_{\hat{f}\hat{g}_1}(t,s) \\
&= \left(\hat{f}^{\hat{2}}+\hat{2}\hat{\times}\hat{b}\hat{\times}\hat{f}-\hat{g}_1^{\hat{2}}+\hat{c}\right)\hat{\times}C_{\hat{f}\hat{g}_1}(t,s) \\
&\quad +\left(-\hat{2}\hat{\times}\hat{f}\hat{\times}\hat{g}_1-\hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}_1\right)\hat{\times}S_{\hat{f}\hat{g}_1}(t,s) \\
&= \left(\hat{b}^{\hat{2}}-\hat{2}\hat{\times}\hat{b}^{\hat{2}}+\hat{b}^{\hat{2}}-\hat{c}+\hat{c}\right)\hat{\times}C_{\hat{f}\hat{g}_1}(t,s) \\
&\quad +\left(\hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}_1-\hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}_1\right)\hat{\times}S_{\hat{f}\hat{g}_1}(t,s) \\
&= 0.
\end{aligned}
$$

Next,

$$\begin{aligned}
D_t^{\alpha\circledast} y_4(t) &= D_t^{\alpha\circledast} S_{\hat{f}\hat{g}_1}(t,s) \\
&= \hat{g}_1 \hat{\times} C_{\hat{f}\hat{g}_1}(t,s) + \hat{f} \hat{\times} S_{\hat{f}\hat{g}_1}(t,s), \\
\left(D_t^{\alpha\circledast}\right)^2 y_4(t) &= D_t^{\alpha\circledast} \left(\hat{g}_1 \hat{\times} C_{\hat{f}\hat{g}_1}(t,s) + \hat{f} \hat{\times} S_{\hat{f}\hat{g}_1}(t,s)\right) \\
&= D_t^{\alpha\circledast} \left(\hat{g}_1 \hat{\times} C_{\hat{f}\hat{g}_1}(t,s)\right) \\
&\quad + D_t^{\alpha\circledast} \left(\hat{f} \hat{\times} S_{\hat{f}\hat{g}_1}(t,s)\right) \\
&= \hat{g}_1 \hat{\times} D_t^{\alpha\circledast} \left(C_{\hat{f}\hat{g}_1}(t,s)\right) \\
&\quad + \hat{f} \hat{\times} D_t^{\alpha\circledast} \left(S_{\hat{f}\hat{g}_1}(t,s)\right) \\
&= \hat{g}_1 \hat{\times} \left(\hat{f} \hat{\times} C_{\hat{f}\hat{g}_1}(t,s) - \hat{g}_1 \hat{\times} S_{\hat{f}\hat{g}_1}(t,s)\right) \\
&\quad + \hat{f} \hat{\times} \left(\hat{g}_1 \hat{\times} C_{\hat{f}\hat{g}_1}(t,s) + \hat{f} \hat{\times} S_{\hat{f}\hat{g}_1}(t,s)\right) \\
&= \left(-\hat{g}_1^{\hat{2}} + \hat{f}^{\hat{2}}\right) \hat{\times} S_{\hat{f}\hat{g}_1}(t,s) \\
&\quad + \hat{2} \hat{\times} \hat{g}_1 \hat{\times} \hat{f} \hat{\times} C_{\hat{f}\hat{g}_1}(t,s).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(D_t^{\alpha\circledast}\right)^2 y_4(t) + \hat{2} \hat{\times} \hat{b} \hat{\times} D_t^{\alpha\circledast} y_4(t) + \hat{c} \hat{\times} y_4(t) \\
&= \left(-\hat{g}_1^{\hat{2}} + \hat{f}^{\hat{2}}\right) \hat{\times} S_{\hat{f}\hat{g}_1}(t,s) + \hat{2} \hat{\times} \hat{g}_1 \hat{\times} \hat{f} \hat{\times} C_{\hat{f}\hat{g}_1}(t,s) \\
&\quad + \hat{2} \hat{\times} \hat{b} \hat{\times} \left(\hat{g}_1 \hat{\times} C_{\hat{f}\hat{g}_1}(t,s) + \hat{f} \hat{\times} S_{\hat{f}\hat{g}_1}(t,s)\right) \\
&\quad + \hat{c} \hat{\times} S_{\hat{f}\hat{g}_1}(t,s)
\end{aligned}$$

$$
\begin{aligned}
&= \left(-\hat{g}_1^{\hat{2}} + \hat{f}^{\hat{2}} + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{f} + \hat{c}\right)\hat{\times}S_{\hat{f}\hat{g}_1}(t,s) \\
&\quad + \left(\hat{2}\hat{\times}\hat{g}_1\hat{\times}\hat{f} + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}_1\right)\hat{\times}C_{\hat{f}\hat{g}_1}(t,s) \\
&= \left(\hat{b}^{\hat{2}} - \hat{c} + \hat{b}^{\hat{2}} - \hat{2}\hat{\times}\hat{b}^{\hat{2}} + \hat{c}\right)\hat{\times}S_{\hat{f}\hat{g}_1}(t,s) \\
&\quad + \left(-\hat{2}\hat{\times}\hat{g}_1\hat{\times}\hat{b} + \hat{2}\hat{\times}\hat{b}\hat{\times}\hat{g}_1\right)\hat{\times}C_{\hat{f}\hat{g}_1}(t,s) \\
&= 0.
\end{aligned}
$$

3. Consider the equation

$$\left(D^{\alpha\circledast}\right)^2 y + \hat{a}^{\hat{2}}\hat{\times}y = 0,$$

where $\hat{a} = \frac{a}{\hat{T}(t)}$, $a \in \mathbb{R}$. We will show that

$$y_1(t) = \cos_{\hat{a}}(t,s) \quad \text{and} \quad y_2(t) = \sin_{\hat{a}}(t,s)$$

are its solutions. We have

$$
\begin{aligned}
D_t^{\alpha\circledast} y_1(t) &= D_t^{\alpha\circledast}\cos_{\hat{a}}(t,s) \\
&= -\hat{a}\hat{\times}\sin_{\hat{a}}(t,s), \\
\left(D_t^{\alpha\circledast}\right)^2 y_1(t) &= D_t^{\alpha\circledast}\left(-\hat{a}\hat{\times}\sin_{\hat{a}}(t,s)\right) \\
&= -\hat{a}\hat{\times}\left(D_t^{\alpha\circledast}\sin_{\hat{a}}(t,s)\right) \\
&= -\hat{a}^{\hat{2}}\hat{\times}\cos_{\hat{a}}(t,s).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left(D_t^{\alpha\circledast}\right)^2 y_1(t) + \hat{a}^{\hat{2}}\hat{\times}y_1(t) &= -\hat{a}^{\hat{2}}\hat{\times}\cos_{\hat{a}}(t,s) \\
&\quad + \hat{a}^{\hat{2}}\hat{\times}\cos_{\hat{a}}(t,s) \\
&= 0.
\end{aligned}
$$

Next,

$$\begin{aligned} D_t^{\alpha\circledast} y_2(t) &= D_t^{\alpha\circledast} \sin_{\hat{a}}(t,s) \\ &= \hat{a}\hat{\times} \cos_{\hat{a}}(t,s), \\ \left(D_t^{\alpha\circledast}\right)^2 y_2(t) &= D_t^{\alpha\circledast}\left(\hat{a}\hat{\times} \cos_{\hat{a}}(t,s)\right) \\ &= \hat{a}\hat{\times}\left(D_t^{\alpha\circledast} \cos_{\hat{a}}(t,s)\right) \\ &= -\hat{a}^{\hat{2}}\hat{\times} \sin_{\hat{a}}(t,s). \end{aligned}$$

Then

$$\begin{aligned} \left(D_t^{\alpha\circledast}\right)^2 y_2(t) + \hat{a}^{\hat{2}}\hat{\times} y_2(t) &= -\hat{a}^{\hat{2}}\hat{\times} \sin_{\hat{a}}(t,s) \\ &\quad +\hat{a}^{\hat{2}}\hat{\times} \sin_{\hat{a}}(t,s) \\ &= 0. \end{aligned}$$

8 Conformable Iso-Integration

Suppose that $\hat{f} : A \to \mathbb{R}$ is an iso-function of the first, second, third, fourth or fifth kind which are continuous on A, $\alpha \in (0,1]$, k_0 and k_1 satisfy (2.1). Let $a, b \in A$, $a < b$.

Definition 8.1. *Define the conformable iso-integral of $\hat{f}$ over $[a,b]$ as follows.*

$$\hat{I}_f^{\alpha}(t,a) = \hat{\int}_a^t \left(\hat{f}(s)\hat{\times} E_0(t,s)\right) \nearrow \left(\hat{T}(s)\hat{k}_0(\alpha,s)\right) \hat{\times} d\hat{s}, \quad t \in [a,b].$$

We will denote

$$\hat{I}_f^{\alpha}(t,a) = \hat{\int}_a^t \hat{f}(s)\hat{\times} d_{\alpha}\hat{s}.$$

Now we will give an explicit representation of the conformable iso-integral. We have

$$\begin{aligned}
\hat{I}_f^{\alpha}(t,s) &= \hat{\int}_a^t \left(\hat{f}(s)\hat{\times}E_0(t,s)\right) \nearrow \left(\hat{T}(s)\hat{k}_0(\alpha,s)\right) \hat{\times}\hat{d}\hat{s} \\
&= \int_a^t \frac{1}{\hat{T}(s)} \frac{\hat{f}(s)\hat{T}(s)E_0(t,s)}{\left(\hat{T}(s)\right)^2 \hat{k}_0(\alpha,s)} \left(\hat{T}(s)\right)^2 d\hat{s} \\
&= \int_a^t \frac{\hat{f}(s)E_0(t,s)}{\hat{k}_0(\alpha,s)} \frac{\hat{T}(s) - s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2} ds.
\end{aligned}$$

Theorem 8.2. *We have*

$$D_t^{\alpha\circledast}\left(\hat{\int}_a^t \hat{f}(s)\hat{\times}\hat{d}_\alpha\hat{s}\right) = \hat{f}(t).$$

Proof. Note that

$$\begin{aligned}
\left(\hat{I}_f^{\alpha}(t,a)\right)_t^{\circledast} &= \frac{1}{\hat{T}(t)} \frac{d\hat{I}_f^{\alpha}(t,a)}{d\hat{t}} \\
&= \frac{\hat{T}(t)}{\hat{T}(t) - t\hat{T}'(t)} \left(\hat{I}_f^{\alpha}(t,a)\right)_t' \\
&= \frac{\hat{T}(t)}{\hat{T}(t) - t\hat{T}'(t)} \left(\frac{\hat{f}(t)E_0(t,t)}{\hat{T}(t)\hat{k}_0(\alpha,t)} \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right. \\
&\quad \left. + \int_a^t \frac{\hat{f}(s)E_{0t}'(t,s)}{\hat{k}_0(\alpha,s)} \frac{\hat{T}(s) - s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2} ds \right) \\
&= \frac{\hat{f}(t)}{\hat{T}(t)\hat{k}_0(\alpha,t)}
\end{aligned}$$

$$+\frac{\hat{T}(t)}{\hat{T}(t)-t\hat{T}'(t)}\int_a^t \frac{\hat{f}(s)}{\hat{k}_0(\alpha,s)}\left(-\frac{\hat{k}_1(\alpha,t)}{\hat{k}_0(\alpha,t)}\left(\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)E_0(t,s)\right)$$
$$\times\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}ds$$
$$= \frac{\hat{f}(t)}{\hat{T}(t)\hat{k}_0(\alpha,t)}-\frac{\hat{k}_1(\alpha,t)}{\hat{k}_0(\alpha,t)}\hat{I}_f^\alpha(t,a).$$

Hence,

$$\begin{aligned}
D_t^{\alpha\circledast}\hat{I}_f^\alpha(t,a) &= \hat{k}_1(\alpha,t)\hat{\times}\hat{I}_f^\alpha(t,a)\\
&\quad +\hat{k}_0(\alpha,t)\hat{\times}\left(\hat{I}_f^\alpha(t,a)\right)_t^{\circledast}\\
&= \hat{k}_1(\alpha,t)\hat{\times}\hat{I}_f^\alpha(t,a)\\
&\quad +\hat{k}_0(\alpha,t)\hat{\times}\frac{\hat{f}(t)}{\hat{k}_0(\alpha,t)\hat{T}(t)}\\
&\quad -\hat{k}_0(\alpha,t)\hat{\times}\left(\frac{\hat{k}_1(\alpha,t)}{\hat{k}_0(\alpha,t)}\hat{I}_f^\alpha(t,a)\right)\\
&= \hat{f}(t)+\hat{k}_1(\alpha,t)\hat{\times}\hat{I}_f^\alpha(t,a)\\
&\quad -\hat{k}_1(\alpha,t)\hat{\times}\hat{I}_f^\alpha(t,a)\\
&= \hat{f}(t).
\end{aligned}$$

This completes the proof. □

Theorem 8.3. *Let $\hat{f}$ be iso-differentiable on A. Then*

$$\hat{\int}_a^t D^{\alpha\circledast}\hat{f}(s)\hat{\times}\hat{d}_\alpha\hat{s} = \hat{f}(t)-\hat{f}(a)E_0(t,a), \quad t\in[a,b].$$

Proof. Using the definition for conformable iso-integral, we get

$$\hat{\int}_a^t D^{\alpha\circledast}\hat{f}(s)\hat{\times}\hat{d}_\alpha\hat{s} = \int_a^t \frac{D^{\alpha\circledast}\hat{f}(s)E_0(t,s)}{\hat{k}_0(\alpha,s)}\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}ds$$

$$= \int_a^t \left(\frac{E_0(t,s)}{\hat{k}_0(\alpha,s)}\right)\left(\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}\right)\left(\hat{k}_1(\alpha,s)\hat{\times}\hat{f}(s)\right.$$

$$\left.+\hat{k}_0(\alpha,s)\hat{\times}\left(\frac{\hat{T}(s)}{\hat{T}(s)-s\hat{T}'(s)}\frac{d}{ds}\hat{f}(s)\right)\right)ds$$

$$= \int_a^t \left(\frac{E_0(t,s)}{\hat{k}_0(\alpha,s)}\right)\left(\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}\right)\left(\hat{k}_1(\alpha,s)\hat{\times}\hat{f}(s)\right)ds$$

$$+\int_a^t \left(\frac{E_0(t,s)}{\hat{k}_0(\alpha,s)}\right)\left(\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}\right)k_0(\alpha,s)\frac{\hat{T}(s)}{\hat{T}(s)-s\hat{T}'(s)}\frac{d}{ds}\hat{f}(s)ds$$

$$= \int_a^t \left(\frac{E_0(t,s)}{\hat{k}_0(\alpha,s)}\right)\left(\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}\right)\left(\hat{k}_1(\alpha,s)\hat{\times}\hat{f}(s)\right)ds$$

$$+\int_a^t E_0(t,s)\frac{d}{ds}\hat{f}(s)ds$$

$$= \int_a^t \left(\frac{E_0(t,s)}{\hat{k}_0(\alpha,s)}\right)\left(\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}\right)\left(\hat{k}_1(\alpha,s)\hat{\times}\hat{f}(s)\right)ds$$

$$+E_0(t,t)\hat{f}(t)-\hat{f}(a)E_0(t,a)$$

$$
\begin{aligned}
& - \int_a^t \frac{d}{ds} E_0(t,s)\hat{f}(s)ds \\
= & \int_a^t \left(\frac{E_0(t,s)}{\hat{k}_0(\alpha,s)}\right)\left(\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}\right)\left(\hat{k}_1(\alpha,s)\hat{\times}\hat{f}(s)\right)ds \\
& + \hat{f}(t) - \hat{f}(a)E_0(t,a) \\
& - \int_a^t \frac{\hat{k}_1(\alpha,s)}{\hat{k}_0(\alpha,s)}\left(\frac{\hat{T}(s)-s\hat{T}'(s)}{\hat{T}(s)}\right)E_0(t,s)\hat{f}(s)ds \\
= & \int_a^t \left(\frac{E_0(t,s)}{\hat{k}_0(\alpha,s)}\right)\left(\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}\right)\left(\hat{k}_1(\alpha,s)\hat{\times}\hat{f}(s)\right)ds \\
& + \hat{f}(t) - \hat{f}(a)E_0(t,a) \\
& - \int_a^t \left(\frac{E_0(t,s)}{\hat{k}_0(\alpha,s)}\right)\left(\frac{\hat{T}(s)-s\hat{T}'(s)}{\left(\hat{T}(s)\right)^2}\right)\left(\hat{k}_1(\alpha,s)\hat{\times}\hat{f}(s)\right)ds \\
= & \ \hat{f}(t) - \hat{f}(a)E_0(t,a), \quad t \in [a,b].
\end{aligned}
$$

This completes the proof. □

Theorem 8.4. *Let $\hat{g} : A \to \mathbb{R}$ be a continuous iso-function of the first, second, third, fourth or fifth kind. Then*

$$
\begin{aligned}
\hat{\int}_a^t \left(\hat{\beta}\hat{\times}\hat{f}(s) + \hat{\gamma}\hat{\times}\hat{g}(s)\right)\hat{\times}\hat{d}_\alpha\hat{s} \ = \ & \hat{\beta}\hat{\times}\hat{\int}_a^t \hat{f}(s)\hat{\times}\hat{d}_\alpha\hat{s} \\
& + \hat{\gamma}\hat{\times}\hat{\int}_a^t \hat{g}(s)\hat{\times}\hat{d}_\alpha\hat{s}
\end{aligned}
$$

for any $\hat{\beta} = \frac{\beta}{\hat{T}(t)}$, $\hat{\gamma} = \frac{\gamma}{\hat{T}(t)}$, $\beta, \gamma \in \mathbb{R}$.

Proof. We have

$$\begin{aligned}
\hat{\int}_a^t \left(\hat{\beta}\hat{\times}\hat{f}(s)+\hat{\gamma}\hat{\times}\hat{g}(s)\right)\hat{\times}\hat{d}_\alpha\hat{s} &= \hat{\int}_a^t \Bigg(\left(\hat{\beta}\hat{\times}\hat{f}(s)+\hat{\gamma}\hat{\times}\hat{g}(s)\right) \\
&\quad \times\hat{\times}E_0(t,s)\Bigg) \nearrow \left(\hat{T}(s)\hat{k}_0(\alpha,s)\right)\hat{\times}\hat{d}\hat{s} \\
&= \hat{\beta}\hat{\times}\hat{\int}_a^t \left(\hat{f}(s)\hat{\times}E_0(t,s)\right) \nearrow \left(\hat{T}(s)\hat{k}_0(\alpha,s)\right)\hat{\times}\hat{d}\hat{s} \\
&\quad +\hat{\gamma}\hat{\times}\hat{\int}_a^t \left(\hat{g}(s)\hat{\times}E_0(t,s)\right) \nearrow \left(\hat{T}(s)\hat{k}_0(\alpha,s)\right)\hat{\times}\hat{d}\hat{s} \\
&= \hat{\beta}\hat{\times}\hat{\int}_a^t \hat{f}(s)\hat{\times}\hat{d}_\alpha\hat{s} \\
&\quad +\hat{\gamma}\hat{\times}\hat{\int}_a^t \hat{g}(s)\hat{\times}\hat{d}_\alpha\hat{s}, \quad t\in[a,b].
\end{aligned}$$

This completes the proof. □

Theorem 8.5 (Integration by Parts). *Let $\hat{f},\hat{g}: A\to\mathbb{R}$ be iso-functions of the first, second, third, fourth or fifth kind which are iso-differentiable on A. Then*

$$\begin{aligned}
\hat{\int}_a^t \hat{f}(s)\hat{\times}D^{\alpha\circledast}\hat{g}(s)\hat{\times}\hat{d}_\alpha\hat{s} &= -\hat{\int}_a^t \hat{g}(s)\hat{\times}D^{\alpha\circledast}\hat{f}(s)\hat{\times}\hat{d}_\alpha\hat{s} \\
&\quad -\hat{\int}_a^t \hat{f}(s)\hat{\times}\hat{g}(s)\hat{\times}\left(\frac{1}{\left(\hat{T}(s)\right)^3}\hat{\times}\hat{T}^{\circledast}(s)-\hat{k}_1(\alpha,s)\right)\hat{\times}\hat{d}_\alpha\hat{s} \\
&\quad +\hat{f}(t)\hat{\times}\hat{g}(t)-\left(\hat{f}(a)\hat{\times}\hat{g}(a)\right)\hat{\times}\hat{E}_0(t,a), \quad t\in[a,b],
\end{aligned}$$

where

$$\hat{E}_0(t,a)=\frac{E_0(t,a)}{\hat{T}(t)}.$$

Proof. Using the rule for the conformable iso-derivative of order α for the iso-product of two iso-functions and then applying Theorem 8.3, we get

$$\begin{aligned}
\hat{\int}_a^t \hat{f}(s)\hat{\times}D^{\alpha\circledast}\hat{g}(s)\hat{\times}\hat{d}_\alpha\hat{s} &= \hat{\int}_a^t D^{\alpha\circledast}\left(\hat{f}(s)\hat{\times}\hat{g}(s)\right)\hat{\times}\hat{d}_\alpha\hat{s} \\
&\quad -\hat{\int}_a^t \hat{g}(s)\hat{\times}D^{\alpha\circledast}\hat{f}(s)\hat{\times}\hat{d}_\alpha\hat{s} \\
&\quad -\hat{\int}_a^t \hat{f}(s)\hat{\times}\hat{g}(s)\hat{\times}\left(\frac{1}{\left(\hat{T}(s)\right)^3}\hat{\times}\hat{T}^{\circledast}(s) - \hat{k}_1(\alpha,s)\right)\hat{\times}\hat{d}_\alpha\hat{s} \\
= \; &-\hat{\int}_a^t \hat{g}(s)\hat{\times}D^{\alpha\circledast}\hat{f}(s)\hat{\times}\hat{d}_\alpha\hat{s} \\
&\quad -\hat{\int}_a^t \hat{f}(s)\hat{\times}\hat{g}(s)\hat{\times}\left(\frac{1}{\left(\hat{T}(s)\right)^3}\hat{\times}\hat{T}^{\circledast}(s) - \hat{k}_1(\alpha,s)\right)\hat{\times}\hat{d}_\alpha\hat{s} \\
&\quad +\hat{f}(t)\hat{\times}\hat{g}(t) - \left(\hat{f}(a)\hat{\times}\hat{g}(a)\right)\hat{\times}\hat{E}_0(t,a), \quad t\in[a,b].
\end{aligned}$$

This completes the proof. □

9 Applications to First Order Conformable Iso-Differential Equations

Let $\hat{f},\hat{g}: A\to\mathbb{R}$ be iso-functions of the first, second, third, fourth or fifth kind which are continuous on A. Let also, $t_0\in A$ and $\hat{y}_0=\frac{y_0}{\hat{T}(t)}$, $y_0\in\mathbb{R}$. Consider the IVP

$$D^{\alpha\circledast}y = \hat{f}(t)\hat{\times}y+\hat{g}(t), \quad t>t_0, \tag{9.1}$$

$$y(t_0)=\hat{y}_0. \tag{9.2}$$

Theorem 9.1. *The IVP* (9.1), (9.2) *has unique solution represented in the form*

$$y(t) = \hat{T}(t_0)\hat{y}_0\hat{E}_{-\hat{h}+\hat{k}_1}(t,t_0)$$

$$+\frac{1}{\hat{T}(t_0)}\hat{\int}_{t_0}^{t}\hat{g}(s)\hat{\times}E_{\hat{h}}(s,t_0)\hat{\times}\hat{d}_\alpha\hat{s}, \quad t>t_0,$$

where

$$\hat{h}(t) = -\hat{f}(t) - \frac{1}{\left(\hat{T}(t)\right)^3}\hat{\times}\hat{T}^{P\circledast}(t) + \hat{k}_1(\alpha,t), \quad t>t_0.$$

Proof. We rewrite the equation (9.1) in the form

$$D^{\alpha\circledast}y = -\left(\hat{h}(t) + \frac{1}{\left(\hat{T}(t)\right)^3}\hat{\times}\hat{T}^{\circledast}(t) - \hat{k}_1(\alpha,t)\right)\hat{\times}y + \hat{g}(t),$$

$t>t_0$. We iso-multiply the last equation by $E_{\hat{h}}(t,t_0)$ and we get

$$E_{\hat{h}}(t,t_0)\hat{\times}D^{\alpha\circledast}y = -\left(\hat{h}(t) + \frac{1}{\left(\hat{T}(t)\right)^3}\hat{\times}\hat{T}^{\circledast}(t) - \hat{k}_1(\alpha,t)\right)\hat{\times}y\hat{\times}E_{\hat{h}}(t,t_0)$$

$$+\hat{g}(t)\hat{\times}E_{\hat{h}}(t,t_0), \quad t>t_0.$$

From here,

$$D^{\alpha\circledast}\left(E_{\hat{h}}(\cdot,t_0)\hat{\times}y\right)(t) = \hat{g}(t)\hat{\times}E_{\hat{h}}(t,t_0), \quad t>t_0.$$

Hence,

$$\hat{\int}_{t_0}^{t}D^{\alpha\circledast}\left(E_{\hat{h}}(\cdot,t_0)\hat{\times}y\right)(s)\hat{\times}\hat{d}_\alpha\hat{s} = \hat{\int}_{t_0}^{t}\hat{g}(s)\hat{\times}E_{\hat{h}}(s,t_0)\hat{\times}\hat{d}_\alpha\hat{s}, \quad t>t_0,$$

or

$$E_{\hat{h}}(t,t_0)\hat{\times}y(t) - \left(E_{\hat{h}}(t_0,t_0)\hat{\times}\hat{y}(t_0)\right)E_0(t,t_0)$$

$$= \hat{\int}_{t_0}^{t}\hat{g}(s)\hat{\times}E_{\hat{h}}(s,t_0)\hat{\times}\hat{d}_\alpha\hat{s}.$$

Hence,

$$E_{\hat{h}}(t,t_0)\hat{T}(t)y(t) = \hat{T}(t_0)\hat{y}_0 E_0(t,t_0) + \hat{\int}_{t_0}^{t} \hat{g}(s)\hat{\times}E_{\hat{h}}(s,t_0)\hat{\times}\hat{d}_\alpha\hat{s},$$

or

$$\begin{aligned} y(t) &= \hat{T}(t_0)\hat{y}_0\hat{E}_{0\hat{\ominus}\hat{h}}(t,t_0) \\ &\quad + \frac{1}{\hat{T}(t)}\hat{\int}_{t_0}^{t} \hat{g}(s)\hat{\times}E_{\hat{h}}(s,t)\hat{\times}\hat{d}_\alpha\hat{s} \\ &= \hat{T}(t_0)\hat{y}_0\hat{E}_{-\hat{h}+\hat{k}_1}(t,t_0) \\ &\quad + \frac{1}{\hat{T}(t)}\hat{\int}_{t_0}^{t} \hat{g}(s)\hat{\times}E_{\hat{h}}(s,t)\hat{\times}\hat{d}_\alpha\hat{s}, \quad t > t_0. \end{aligned}$$

This completes the proof. □

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INHOMOGENEOUS VISCOUS FLUID COSMOLOGICAL MODEL OF THE UNIVERSE WITH EFFECTIVE EQUATION OF STATE AND SCALAR FIELD

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Abstract

In this study we considered inhomogeneous viscous fluid in flat Friedmann-Robertson-Walker (FRW) universe and also assumed the bulk viscosity as a linear combination of two terms: one is constant and other is proportional to scalar expansion θ. In this framework, by using holographic dark energy model with new infrared cut off introduced by Granda and Oliveros [Physics Letters B, **669**, 275-277 (2008)], we obtained the analytical solutions of the field equations for scale factor $a(t)$, scalar field potential $V(\phi)$ with modified equation of state (EoS) of the form $p = \omega(\rho)\rho - 3\zeta(H)H + \Lambda(t)$, where $\omega(\rho) = A_0\rho^{\alpha_1 - 1} - 1, \zeta(H) = \zeta_0 + \zeta_1 H$ and $\Lambda(t) = \Lambda_0 + \Lambda_1 H$. The final results are compared with the previously known results.

Keywords: Inhomogeneous viscous fluid; dark energy; effective equation of state; scalar field.

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1 Introduction

The astronomical observations indicate that the expansion of our universe is accelerates [1]-[4]. The cosmological observations have provided increasingly convincing evidence that our universe is undergoing a late-time cosmic acceleration expansion. In the last decade lots of research work on its possible mechanism, such as extended gravity [5], modifying equation of state or dark energy models, are to explain the cosmic acceleration expansion observed. To overcome the drawback of hydrodynamical instability, a linear EoS of a more general form $p = \alpha(\rho - p_0)$, was proposed by Babichev et al. [6] and this form incorporated into cosmological model can describe the hydrodynamically stable dark energy behaviors.

The dissipative effect in the fluid is always due to shear and bulk viscosity characterized by shear viscosity parameter η and bulk viscosity parameter ζ. In the study of cosmology, the shear viscosity disappears in the Friedmann-Robertson-Walker's (FRW) isotropic and homogeneous framework with the largest spherical symmetry. Hence only bulk viscosity could play a role in the realistic models. Its effects can be shown by an added correction term with the minus sign to cosmic pressure $\bar{p} = p - 3\zeta H$. This composite formula inspires the trial on the connection between bulk viscosity of the cosmic fluid and dark components. In Refs [7]-[16] the bulk viscosity in cosmology has been studied in various aspects. The dark energy universe with a viscous dark fluid was studied in [17]-[24]. Similarly Nojiri and Odintsov [25]-[27] and Capozziello et al.[28] studied the bulk viscosity in various aspects.

Brevik et al. [29] investigated viscous cosmology in the early universe and in the late universe, containing current acceleration and the possible future singularities and also investigated how one may even unify inflationary and late-time acceleration.

The influence of bulk viscosity in the cosmic fluid plays an important role in the Big Rip phenomenon, namely when the singularity of the universe appears in the future [30]-[33] or in the type II, type III and type IV Rips, [34]-[36] which correspond to the cases where one or more of the relevant physical quantities go to infinity, in different specific ways, at some finite value of time in the future. Time-dependent bulk viscosity [37], a linear EoS [6], [38] and the Hubble parameter dependent EoS [25] were considered

in the study of the dark energy physics. Brevik et al. [39] presented dark energy model with an inhomogeneous EoS of the form $p = \omega(t)\rho + \Lambda(t)$ where $\omega(t)$ and $\Lambda(t)$ are time dependent parameters. Meng et al. [40] assumed the EoS $p = (\gamma - 1)\rho + p_0$, where γ and p_0 are two parameters and obtained an exact solution of the scale factor. Khadekar et al. [41] has considered a universe media as a bulk viscosity described by inhomogeneous EoS of the form $p = (\gamma - 1)\rho + \Lambda(t)$ and discussed the acceleration expansion of the universe evolution and future singularities for FRW model.

Recently Khadekar and Deepti Raut [42] have presented two viscous models of non-perfect fluid by avoiding the introduction of exotic dark energy in terms of deceleration parameter q has a viscosity of the form $\zeta = \zeta_0 + (\zeta_1 - \zeta_2 q)H$ and the other model in quadratic form of H of the type $\zeta = \zeta_0 + \zeta_1 H + \zeta_2 H^2$ and obtained the solutions of the field equations by using inhomogeneous equation of state.

The EoS of inhomogeneous viscous fluids in a flat FRW space time is given by [25]-[26], and [28]

$$p = \omega(\rho)\rho + B\left[\rho, a(t), H, \dot{H}\right],$$

where the EoS parameter $\omega(\rho)$, may depends on the energy density, and the bulk viscosity $B\left[\rho, a(t), H, \dot{H}, ...\right]$ is a general function of the fluid energy density, the scale factor, the Hubble parameter and its derivatives. Myrzakul et al. [43] has considered simple formulation of such equation as $p = \omega(\rho)\rho - 3H\zeta(H)$, where $\zeta(H)$ is the bulk viscosity and it depends on the Hubble parameter H only. Myrzakul et al. [43], Nojiri and Odintsov [44] and Astashenok et al.[45] has considered the thermodynamic parameter $\omega(\rho, t)$ as a function of both energy density of the fluid and cosmic time in the form of $\omega(\rho, t) = A_0\rho^{\alpha-1} - 1$, where $A_0 \neq 0$ and $(\alpha \neq 1)$ are constants. Khadekar and Rupali Talole [46] has proposed the FRW model with effective EoS of the form $p = \omega(\rho)\rho + \Lambda(t)$ with bulk viscosity ζ and cosmological constant Λ as a linear combination of two terms and discussed the current accelerating universe expansion.

Scalar fields naturally arise in particle physics including super symmetric field theories and string/M theory [47] and these can act as candidates for dark energy. Scalar fields are assumed to play a fundamental role in cosmology, where one of the first major mechanism for which scalar fields

are thought to be responsible is the inflationary scenario [48]-[49]. Meng and Ren [50] obtained the equivalence of the modified EoS and scalar field model and also observed that with the modified EoS, the scale factor $a(t)$ and scalar field potential $V(\phi)$ all have possessed analytical solutions.

Inspired by the holographic principle [51] proposed "Holographic dark energy (HDE)" having density $\rho_\Lambda = 3c^2M_p^2L^{-2}$, where L is the infrared cut-off. Quite a good number of works have been carried out still date and some of those are in in Refs [52]. Granda and Oliveros [53] considered a special form of HDE dubbed as "extended holographic Ricci dark energy" (EHRDE), whose density is given by $\rho_\Lambda = 3M_p^2(\alpha H^2 + \beta\dot{H})$, where the upper dot represents derivative with respect to cosmic time t, M_p^2 is the reduced Planck mass, α and β are constants.

Chattopadhyay [54] studied, the modified Chaplygin gas (MCG) with viscous EHRDE having energy density of the form $\rho_\Lambda = 3M_p^2(\alpha H^2+\beta\dot{H})$, by choosing bulk viscosity coefficient $\zeta = \zeta_0+\zeta_1 H+\zeta_2(\dot{H}+H^2)$, where ζ_0, ζ_1 and ζ_2 are constant coefficients and H is the Hubble parameter. Chattopadhyay [54] also studied the viscous EHRDE in flat FRW universe based on Israel-Stewart approach.

With the motivation of the above work in this paper we solve FRW equations with modified EoS of the form $p = \omega(\rho)\rho - 3\zeta(H)H + \Lambda(t)$, with bulk viscosity of the type $\zeta = \zeta_0 + \zeta_1 H$ by using EHRDE and obtained the solution of the field equation.

The paper is organized as follows: In section 2, we solve the dynamical equation of the scale factor and obtain the solution with inhomogeneous EoS parameter $\omega(\rho) = A_0\rho^{\alpha_1-1} - 1$ and obtain the physical parameters namely scale factor a, Hubble parameter H and density parameter ρ. We discuss the special cases namely $T_1 \to \infty$, $T_2 \to \infty$ & $T \to \infty$ and again obtain the solution for the scale factor. In Section 3, we discuss the future singularities of the model and sound speed. In section 4 we shall study cosmological dynamics of viscous EHRDE by introducing scalar field ϕ and scalar potential $V(\phi)$. Section 5, deals with conclusion and discussion.

2 FRW Universe with Holographic Viscous Dark Energy

We consider the flat FRW universe with line element:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)\right], \tag{1}$$

where $a(t)$ is a scale factor with cosmic time t.
We assume EHRDE [53]

$$\rho_\Lambda = 3M_p^2(\alpha H^2 + \beta\dot{H}), \tag{2}$$

where $H = \dot{a}/a$ is the Hubble parameter and α and β are constants.
Einstein field equations are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\delta^\mu_\nu R = 8\pi G T_{\mu\nu}, \tag{3}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar and $T_{\mu\nu}$ is the energy momentum tensor.
The energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{4}$$

where ρ and p are energy density and pressure respectively.
The conservation equation for energy is given by

$$\dot{\rho} + (\rho + p)\,\theta = 0, \tag{5}$$

where $\theta = 3H$,scalar expansion.
We obtain the Friedmann equations as follows

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho \;\Rightarrow\; H^2 = \frac{8\pi G}{3}\rho, \tag{6}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \;\Rightarrow\; \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p), \tag{7}$$

where $(\cdot)$ represent derivative with respect to t.
We consider the EoS for inhomogeneous viscous fluid of the form

$$p = \omega(\rho)\rho - 3\zeta(H)H + \Lambda(t), \tag{8}$$

where $\omega(\rho)$ is EoS parameter explicitly depending on the energy density of the fluid, i.e.,

$$\omega(\rho) = A_0\rho^{\alpha_1 - 1} - 1, \tag{9}$$

where A_0 and α_1 are constants.
We consider the bulk viscosity coefficient $\zeta(H)$ in the form of

$$\zeta(H) = \zeta_0 + \zeta_1 H, \tag{10}$$

where ζ_0 and ζ_1 are two constants conventionally and $\Lambda(t)$ is the time dependent cosmological constant, which can be expressed as

$$\Lambda(t) = \Lambda_0 + \Lambda_1 H, \tag{11}$$

where Λ_0 and Λ_1 are constants.
The energy conservation law for inhomogeneous viscous fluid in the term of EHRDE ρ_Λ is given by

$$\dot{\rho_\Lambda} + 3H\rho_\Lambda\left[\omega(\rho_\Lambda) + 1\right] = 9H^2\zeta(H) - 3H\Lambda(t). \tag{12}$$

To solve Eq. (12) we consider $\alpha_1 = \frac{1}{2}$ in Eq. (9), $\beta = 0$ and $M_p^2 = 1$ in Eq.(12), we get the dynamical equation which determine the scale factor $a(t)$.

$$\dot{H} = -\frac{3}{2}\left(-\frac{\zeta_1}{\alpha}\right)H^2 + \left[-\left(A_0\sqrt{3\alpha} - 3\zeta_0 + \Lambda_1\right)\frac{1}{2\alpha}\right]H + \left(-\frac{1}{2\alpha}\Lambda_0\right) \tag{13}$$

Hence the dynamical Eq. (13) of the scale factor $a(t)$ can be written as

$$\dot{H} = -\frac{3}{2}\bar{\gamma}H^2 + \frac{1}{T_1}H + \frac{1}{{T_2}^2} \Rightarrow \frac{\ddot{a}}{a} = -\left(\frac{3\bar{\gamma} - 2}{2}\right)\frac{\dot{a}^2}{a^2} + \frac{1}{T_1}\frac{\dot{a}}{a} + \frac{1}{{T_2}^2}, \tag{14}$$

where

$$\bar{\gamma} = -\frac{\zeta_1}{\alpha}, \quad \frac{1}{T_1} = -\left[A_0\sqrt{3\alpha} - 3\zeta_0 + \Lambda_1\right]\frac{1}{2\alpha},$$

$$\frac{1}{{T_2}^2} = -\frac{1}{2\alpha}\Lambda_0, \quad T = \frac{T_1}{\sqrt{1+6\bar{\gamma}\left(\frac{T_1}{T_2}\right)^2}}. \tag{15}$$

Here T_1 and T_2 are criteria to determine whether we should concern ζ and Λ. If $T_1 >> t$, the cosmic time scale, the effect of ζ can be neglected, and if $T_2 >> t$, the effect of Λ can be neglected likewise.
By assuming initial conditions as $a(t_0) = a_0$, $\theta(t_0) = \theta_0$ with $\bar{\gamma} \neq 0$, we obtain the solutions for the cosmological parameters like Hubble parameter, the scale factor and the energy density as follows

$$\frac{\dot{a}}{a} = \frac{1}{3\bar{\gamma}}X_1, \tag{16}$$

where $X_1 = \left[\frac{(1+\theta_0\bar{\gamma}T-T/T_1)\left(\frac{1}{T_1}+\frac{1}{T}\right)e^{(t-t_0)/T}-(1-\theta_0\bar{\gamma}T+T/T_1)\left(\frac{1}{T}-\frac{1}{T_1}\right)}{(1+\theta_0\bar{\gamma}T-T/T_1)e^{(t-t_0)/T}+(1-\theta_0\bar{\gamma}T+T/T_1)}\right]$.

$$a(t) = a_0\,(X_2)^{2/3\bar{\gamma}}, \tag{17}$$

where $X_2 = \left\{\frac{1}{2}\,(1+\bar{\gamma}\theta_0 - T/T_1)\,e^{\left[\frac{t-t_0}{2}\left(\frac{1}{T}+\frac{1}{T_1}\right)\right]} + \frac{1}{2}\,(1-\bar{\gamma}\theta_0 T + T/T_1)\,e^{\left[-\frac{t-t_0}{2}\left(\frac{1}{T}-\frac{1}{T_1}\right)\right]}\right\}$.

$$\rho = \frac{1}{3\kappa^2\bar{\gamma}^2}\left[\frac{(1+\theta_0\bar{\gamma}T-T/T_1)\left(\frac{1}{T_1}+\frac{1}{T}\right)e^{(t-t_0)/T}-(1-\theta_0\bar{\gamma}T+T/T_1)\left(\frac{1}{T}-\frac{1}{T_1}\right)}{(1+\theta_0\bar{\gamma}T-T/T_1)\,e^{(t-t_0)/T}+(1-\theta_0\bar{\gamma}T+T/T_1)}\right]^2. \tag{18}$$

Similarly with initial conditions $a(t_0) = a_0$, $\theta(t_0) = \theta_0$, and if $\bar{\gamma} = 0$, we obtain the solutions for the Hubble parameter, the scale factor and the energy density as follows

$$\frac{\dot{a}}{a} = \left(\frac{\theta_0}{3} + \frac{T_1}{{T_2}^2}\right)e^{(t-t_0)/T_1} - \frac{T_1}{{T_2}^2}, \tag{19}$$

$$a(t) = a_0 exp\left[\left(\frac{\theta_0 T_1}{3} + \frac{{T_1}^2}{{T_2}^2}\right)\left(e^{(t-t_0)/T_1}-1\right) - \frac{T_1}{{T_2}^2}\,(t-t_0)\right], \tag{20}$$

$$\rho(t) = \frac{3}{\kappa^2}\left[\frac{1}{3}\theta_0 e^{(t-t_0)/T_1} + \frac{T_1}{{T_2}^2}\left(e^{(t-t_0)/T_1}-1\right)\right]^2. \tag{21}$$

3 Special cases

3.1 For $\bar{\gamma} \neq 0$

We solve Eq.(14) for the special cases as follows:
When $T_2 \to \infty$ i.e. $\Lambda_0 = 0$ then from Eq. (14), we have the solution of the form

$$a(t) = a_0 \left[1 + \frac{1}{2}\bar{\gamma}\theta_0 T \left(e^{(t-t_0)/T} - 1\right)\right]^{2/3\bar{\gamma}}. \tag{22}$$

When $T_1 \to \infty$ i.e. $\Lambda_1 = 3\zeta_0 - A_0\sqrt{3\alpha}$ then we get,

$$a(t) = a_0 \left[cosh\frac{t-t_0}{2T} + \bar{\gamma}\theta_0 T sinh\frac{t-t_0}{2T}\right]^{2/3\bar{\gamma}}. \tag{23}$$

Similarly when $T \to \infty$, then we have

$$a(t) = a_0 \left[1 + \frac{1}{2}\bar{\gamma}\theta_0 \left(t - t_0\right)\right]^{2/3\bar{\gamma}}. \tag{24}$$

3.2 For $\bar{\gamma} = 0$

When $\bar{\gamma} = 0$ then from the Eq. (14), we have the solution of the form

$$a(t) = a_0 exp\left[\left(\frac{1}{3}\theta_0 T_1 + \frac{{T_1}^2}{{T_2}^2}\right)\left(e^{(t-t_0)/T_1} - 1\right) - \frac{T_1(t-t_0)}{{T_2}^2}\right], \tag{25}$$

when $T_2 \to \infty$ then from Eq. (14), we have

$$a(t) = a_0 exp\left[\frac{1}{3}\theta T_1 \left(e^{(t-t_0)/T_1} - 1\right)\right], \tag{26}$$

when $T_1 \to \infty$ then from the equation (14), we have

$$a(t) = a_0 exp\left[\frac{1}{3}\theta_0 \left(t - t_0\right) + \frac{\left(t-t_0\right)^2}{2{T_2}^2}\right], \tag{27}$$

Similarly when $T \to \infty$, we have

$$a(t) = a_0 e^{\theta_0(t-t_0)/3}, \tag{28}$$

which corresponds to the de-Sitter universe with constant scalar expansion.

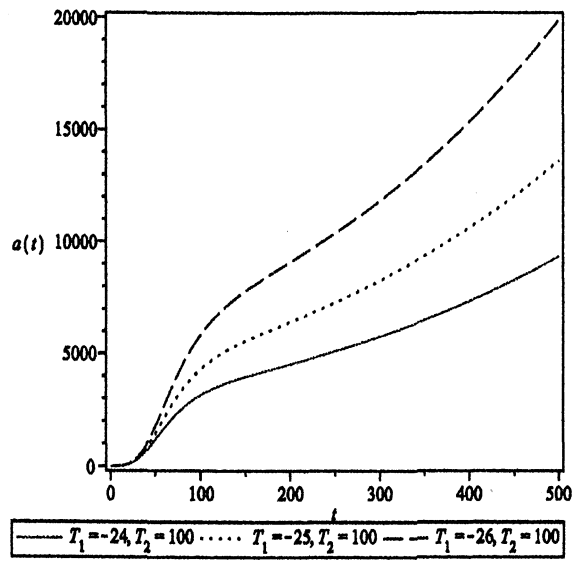

Figure 1: Time evolution of the scale factor $a(t)$, for $\bar{\gamma} = 0$

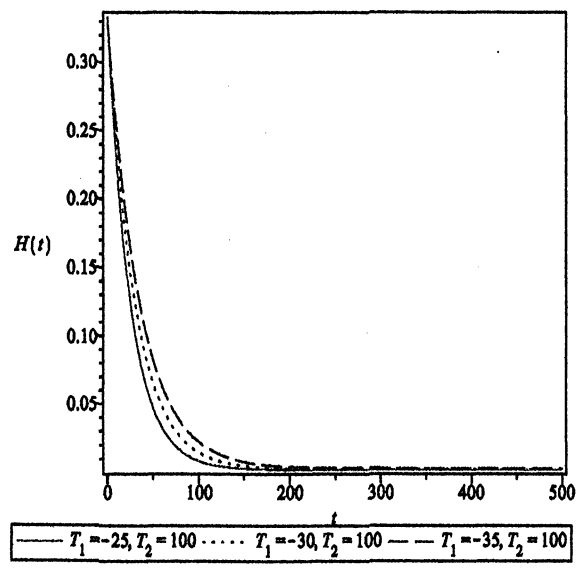

Figure 2: Time evolution of the Hubble parameter $H(t)$, for $\tilde{\gamma} = 0$

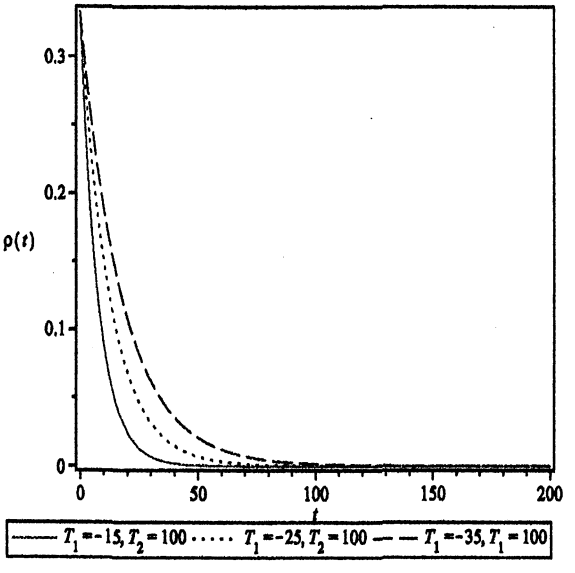

Figure 3: Time evolution of the energy density $\rho(t)$, for $\bar{\gamma} = 0$

3.3 Future Singularities

The future singularities were studied in various contents by [30]-[33] and are classified as follows [34]-[36]:

- Type I (Big Rip): For $t \to t_s$, $a \to \infty$, $\rho \to \infty$, and $|p| \to \infty$,
- Type II (Sudden): For $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$, and $|p| \to \infty$,
- Type III : For $t \to t_s$, $a \to a_s$, $\rho \to \infty$, and $|p| \to \infty$,
- Type IV : For $t \to t_s$, $a \to a_s$, $\rho \to 0$, $|p| \to 0$, and derivatives of H diverge.

In the following cases we discuss that different choices of the parameter may lead to three fates of the universe evolution like no future singularity, big rip, or the Type-III singularity.
Case I: $\bar{\gamma} < 0$
Eq.(2) or Eq.(16) gives

$$\sqrt{\rho} \propto \frac{\dot{a}}{a}. \tag{29}$$

We observed that, if the denominator of X_1 is zero at a finite time t,

$$(1 + \theta_0 \bar{\gamma} T - T/T_1)\, e^{(t-t_0)/T} + (1 - \theta_0 \bar{\gamma} T + T/T_1) = 0, \tag{30}$$

then $a \to \infty$, $\rho \to \infty$, and so the big rip occurs.
Hence from Eq. (30), the solution for time t is

$$t_s = T ln \left(-\frac{1 - \theta_0 \bar{\gamma} T + T/T_1}{1 + \theta_0 \bar{\gamma} T - T/T_1} \right) + t_0. \tag{31}$$

From Eq.(17), (18) and (19), it is observed that in the above result T is complex number and in the following result we assume T is real.
If we want to prevent the big rip, there should be no real solution for $t > t_0$ so

$$\left(-\frac{1 - \theta_0 \bar{\gamma} T + T/T_1}{1 + \theta_0 \bar{\gamma} T - T/T_1} \right) < 1, \tag{32}$$

which is equivalent to

$$1 + \theta_0 \bar{\gamma} T - T/T_1 > 0. \tag{33}$$

After substituting the value of T from Eq. (15) in the above inequality, we get

$$1+\frac{\theta_0\bar{\gamma}T_1}{\sqrt{1+6\bar{\gamma}\left(T_1/T_2\right)^2}}-\frac{1}{\sqrt{1+6\bar{\gamma}\left(T_1/T_2\right)^2}}>0. \tag{34}$$

From this inequality, we obtain the following conclusions:

- $\Lambda_0 < 0$, i.e. ${T_2}^2 > 0$: Inequality (34) is always unsatisfied, so there will be a big rip at time t_s.
- $\Lambda_0 > 0$, i.e. ${T_2}^2 < 0$: Inequality (34) is not always satisfied. If it is unsatisfied so there will be a big rip at time t_s. If it is satisfied, there is no future singularity.

Case II: $\bar{\gamma} > 0$
If in Eq. (17), $X_2 = 0$, then $a \to 0$, $\rho \to \infty$, so the type III singularity occurs. Following the steps as before, we observe that

- $\Lambda_0 < 0$, i.e. ${T_2}^2 > 0$: Inequality (34) is always satisfied, so there is no future singularity.
- $\Lambda_0 > 0$, i.e. ${T_2}^2 < 0$: Inequality (34) is not always satisfied. If it is unsatisfied so there will be type III singularity at time t_s.

Case III: $\bar{\gamma} = 0$
For $\bar{\gamma} = 0$, we have

$$\frac{\dot{a}}{a}=\frac{1}{3}\theta_0 exp\left[\frac{t-t_0}{T_1}\right]+\frac{T_1^2}{T_2^2}\left[exp\left(\frac{t-t_0}{T_1}\right)-1\right]. \tag{35}$$

Note that the solution $a(t)$ for this case has not possessed the future singularity.

3.4 Sound Speed

The EoS for holographic energy density is given by

$$p_\Lambda = (\bar{\gamma} - 1)\rho_\Lambda - \frac{2\sqrt{\alpha}}{\sqrt{3}T_1}\frac{1}{\sqrt{\rho_\Lambda}} - \frac{2\alpha}{T_2^2}\frac{1}{\rho_\Lambda}. \tag{36}$$

The square of sound speed is

$$c_s^2 = \frac{\partial p_\Lambda}{\partial \rho_\Lambda} = \bar{\gamma} - 1 - \frac{\sqrt{\alpha}}{\sqrt{3}T_1}\frac{1}{\sqrt{\rho_\Lambda}}. \tag{37}$$

It is observed that when $\bar{\gamma} = 0$, T_1 should be negative if the sound speed is a real number. It is also seen that with assumption of $T_1 < 0$, the density approaches to a constant in a late time and so sound speed approaches to a constant.
The EoS parameter with holographic energy density is

$$\omega = \frac{p_\Lambda}{\rho_\Lambda} = \bar{\gamma} - 1 - \frac{2\sqrt{\alpha}}{\sqrt{3}T_1}\frac{1}{\sqrt{\rho_\Lambda}} - \frac{2\alpha}{T_2^2}\frac{1}{\rho_\Lambda}. \tag{38}$$

It is observed that ω approaches to constant in late time.

4 Reconstruction of Scalar Field Potential

By following the work of Chattopadhyay [54], the energy density of EHRDE is

$$\rho_\Lambda = 3(\alpha H^2 + \beta \dot{H}), \tag{39}$$

where $M_p^2 = 1$.
Now, by considering $\rho = \rho_\Lambda$ in Eq.(6), we obtain the solution for Hubble parameter H as a function of scale factor

$$H^2 = H_0^2 a^{\frac{2(1-\alpha)}{\beta}} \tag{40}$$

Since $H = \dot{a}/a$, by solving Eq. (40), we obtain the solution for the scale factor as a function of cosmic time t

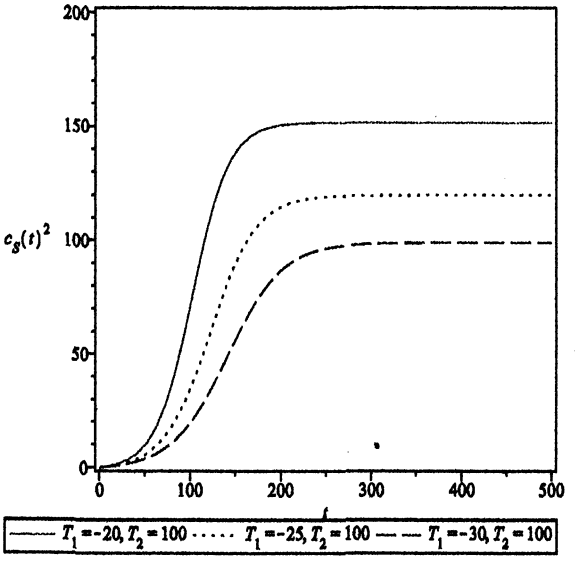

Figure 4: Time evolution of the square of sound speed ${c_s}^2$, for $\bar{\gamma} = 0$

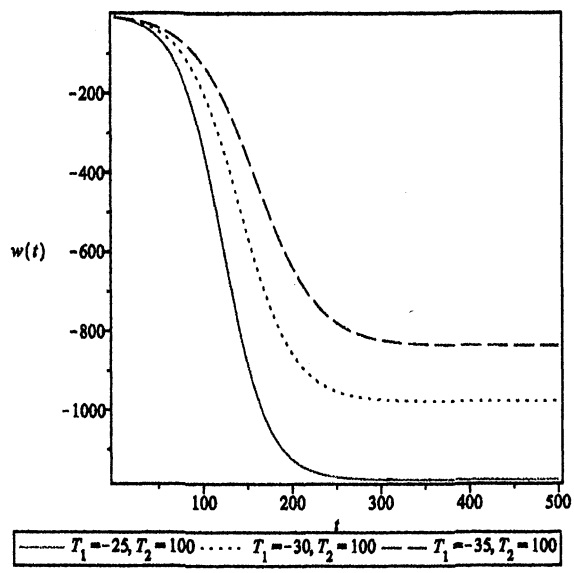

Figure 5: Time evolution of the EoS parameter $\omega(t)$

$$a(t) = \left[1 + \frac{H_0 t}{\beta}(\alpha - 1)\right]^{\frac{\beta}{\alpha-1}}. \tag{41}$$

Similarly, time evolution of Hubble parameter is

$$\dot{H} = H_0^2 \frac{1-\alpha}{\beta} a^{\frac{2(1-\alpha)}{\beta}}. \tag{42}$$

We assume that the isotropic pressure of the cosmological fluid obey EoS for inhomogeneous viscous fluid

$$p_{eff} = \omega(\rho_\Lambda)\rho_\Lambda - 3\zeta(H)H + \Lambda(t), \tag{43}$$

$$p_{eff} = (A_0 \rho_\Lambda^{\alpha_1 - 1} - 1)\rho_\Lambda - 3(\zeta_0 + \zeta_1 H)H + (\Lambda_0 + \Lambda_1 H). \tag{44}$$

So we can obtain the EoS parameter w_{eff} as

$$w_{eff} = A_0 \rho_\Lambda^{\alpha_1} - 1 - \frac{3(\zeta_0 H + \zeta_1 H^2)}{\rho_\Lambda} + \frac{\Lambda_0 + \Lambda_1 H}{\rho_\Lambda} \tag{45}$$

Now by considering a correspondence between EHRDE and inhomogeneous viscous fluid, we get the EoS parameter for holographic viscous dark energy

$$w_{eff} = A_0 \left[3(\alpha H^2 + \beta \dot{H})\right]^{\alpha_1} - 1 - \frac{3(\zeta_0 H + \zeta_1 H^2)}{3(\alpha H^2 + \beta \dot{H})} + \frac{\Lambda_0 + \Lambda_1 H}{3(\alpha H^2 + \beta \dot{H})}. \tag{46}$$

In particular case: for $\alpha_1 = 1/2$, we obtain

$$w_{eff} = \frac{A_0}{\left[3(\alpha H^2 + \beta \dot{H})\right]^{1/2}} - 1 - \frac{3(\zeta_0 H + \zeta_1 H^2)}{3(\alpha H^2 + \beta \dot{H})} + \frac{\Lambda_0 + \Lambda_1 H}{3(\alpha H^2 + \beta \dot{H})}, \tag{47}$$

$$w_{eff} = \frac{A_0}{\left(3H_0^2 a^{\frac{2(1-\alpha)}{\beta}}\right)^{1/2}} - 1 - \frac{H_0 a^{\frac{1-\alpha}{\beta}}\left[\zeta_0 - \zeta_1 H_0 a^{\frac{1-\alpha}{\beta}}\right]}{H_0^2 a^{\frac{2(1-\alpha)}{\beta}}} + \frac{\Lambda_0 + \Lambda_1 H}{H_0^2 a^{\frac{2(1-\alpha)}{\beta}}}, \tag{48}$$

$$w_{eff} = \frac{A_0}{3H_0 a^{\frac{(1-\alpha)}{\beta}}} - 1 - \frac{\zeta_0}{H_0 a^{\frac{(1-\alpha)}{\beta}}} - \zeta_1 + \frac{\Lambda_0}{H_0^2 a^{\frac{2(1-\alpha)}{\beta}}} + \frac{\Lambda_1}{H_0 a^{\frac{(1-\alpha)}{\beta}}}, \tag{49}$$

$$w_{eff} = \frac{1}{H_0 a^{\frac{(1-\alpha)}{\beta}}}\left(\frac{A_0}{\sqrt{3}} - \zeta_0 + \Lambda_1\right) - (\zeta_1 + 1) + \frac{\Lambda_0}{H_0^2 a^{\frac{2(1-\alpha)}{\beta}}}. \quad (50)$$

Fundamental role played by scalar fields in evolution of the early universe is well documented in the literature [58]. We discuss the reconstruction of potential by introducing a scalar field ϕ and a self interacting potential $V(\phi)$ with Lagrangian [59].

$$L_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi). \quad (51)$$

The energy density and the pressure of the scalar field are given by

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) = \rho_\Lambda, \quad (52)$$

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) = p_{eff}. \quad (53)$$

Hence

$$\dot{\phi}^2 = 3H^2(1 + w_{eff}), \quad (54)$$

and

$$V(\phi) = \frac{3H^2}{2}(1 - w_{eff}). \quad (55)$$

We reconstruct the scalar field and scalar potential as follows

$$\dot{\phi}^2 = 3H_0^2 a^{\frac{2(1-\alpha)}{\beta}}\left[1 + w_{eff}\right]. \quad (56)$$

Similarly

$$V(\phi) = \frac{3}{2}H_0^2 a^{\frac{2(1-\alpha)}{\beta}}\left[1 - w_{eff}\right]. \quad (57)$$

5 Conclusion and Discussion

In this work, we considered flat FRW universe filled with viscous EHRDE with energy density of the form $\rho_\Lambda = 3(\alpha H^2 + \beta\dot{H})$. We have solved the FRW equations with modified EoS of the form $p = \omega(\rho)\rho - 3\zeta(H)H + \Lambda(t)$, where $\omega(\rho) = A_0\rho^{\alpha_1 - 1} - 1, \zeta(H) = \zeta_0 + \zeta_1 H$ and $\Lambda(t) = \Lambda_0 + \Lambda_1 H$ and discussed the current accelerating universe expansion of a dynamical

equations for $a(t)$. We assumed the bulk viscosity ζ and $\Lambda(t)$ as a linear combination of two terms one is constant and other is proportional to scalar expansion θ.

With this inhomogeneous EoS, a dynamical equation of scale factor is integrable and obtained the exact solution of Einstein field equations for FRW model with EoS parameter $\omega = A_0\rho^{\alpha-1} - 1$ for $\alpha = 1/2$.

This model has additional parameters $\alpha, \zeta_0, \zeta_1, \Lambda_0$ and Λ_1. The inhomogeneous term in EoS describe the dark energy in the universe media with viscosity. We discussed the cases for the value of $\gamma \neq 0$, $\gamma = 0$. For the value of $\alpha_1 = 1/2$, we have obtained the scale factor and energy density for $T_1 \to \infty$, $T_2 \to \infty$ and $T \to \infty$ respectively.

In this case $\bar{\gamma} \neq 0$, we consider two special case: $\Lambda_0 = 0$, $\Lambda_1 = 3\zeta_0 - A_0\sqrt{3\alpha}$ and obtained the scale factor for $T_1 \to \infty$ and $T_2 \to \infty$.

In Section 3, we have shown that the different types of parameters may leads to the fates of the universe evolution: No future singularity, Big rip as Type II singularity, Type III singularity and Type IV singularity. In the the special cases for $\bar{\gamma} < 0$ and $\bar{\gamma} > 0$, we get the analogous results obtained earlier by Meng et al. [40], Khadekar et al. [41] and Khadekar and Deepti Raut [42] in different content.

Fig. 1 is the evolution of the scale factor versus cosmic time t. From this figure it is shown that the constant bulk viscosity has possessed the similar dynamical effect of the cosmological constant Λ. Figs. 2 and 3 shows the variation of Hubble parameter H and energy density ρ against the time t. In this case Hubble parameter and energy density approaches to constant in the late time and which is not zero. Fig. 4 is the graph of square of sound speed ${c_s}^2$ versus cosmic time t relation for $\bar{\gamma} = 0$. From this graph it is shown that sound speed approaches to a constant in the late time. Fig. 5 is the relation between EoS parameter ω versus cosmic time t. We can shows that ω also approaches to a constant in the late times.

Correspondence between pilgrim DE and scalar field models earlier discussed by [55]. Simillarly correspondence between generalized DE and scalar field models of DE has been considered by [56]. In the view of the work mention by the above author we have provided correspondence between EHRDE and inhomogeneous viscous fluid and reconstructed kinetic energy of scalar field and scalar potential.

The scalar field model as a prototype is intensely used to study the dynamical behaviors for the scalar factor in the literature because of its simplicity in formulation and treatment. Thus it is very interesting to note that concerning the bulk viscosity, the modified EoS, the variable cosmological constant model and the scalar field model can be described in one general dynamical equation which determines the scale factor.

6 Acknowledgment

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MULTIPLICATIVE DERIVATION OF GAMMA RINGS

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Abstract

Let $\mathfrak{M}$ and Γ be abelian groups such that $\mathfrak{M}$ is a Γ-ring. A map $d : \mathfrak{M} \to \mathfrak{M}$ is called a *multiplicative derivation* of $\mathfrak{M}$ if its satisfies the property $d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$ for every $x, y \in \mathfrak{M}$ and $\gamma \in \Gamma$. In this paper we establish a condition on $\mathfrak{M}$ that assures that d is a derivation.

Keywords: Multiplicative derivations, Additivity, Gamma rings.

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1 Preliminaries

Let $\mathfrak{M}$ and Γ be two abelian groups. We call $\mathfrak{M}$ a Γ-*ring* if the following conditions are satisfied:

(i) $x\alpha y \in \mathfrak{M}$,

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z,\ x\alpha(y+z) = x\alpha y + x\alpha z$,

(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$,

(iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

for all $x, y, z \in \mathfrak{M}$ and $\alpha, \beta \in \Gamma$.

A nonzero element $1 \in \mathfrak{M}$ is called a multiplicative γ-identity of $\mathfrak{M}$ or γ-*unity element* (for some $\gamma \in \Gamma$) if $1\gamma x = x\gamma 1 = x$ for all $x \in \mathfrak{M}$. A nonzero element $e_1 \in \mathfrak{M}$ is called a γ_1-*idempotent* (for some $\gamma_1 \in \Gamma$) if $e_1\gamma_1 e_1 = e_1$ and a *nontrivial* γ_1-*idempotent* if it is a γ_1-idempotent different from multiplicative γ_1-identity element of $\mathfrak{M}$.

Let Γ and $\mathfrak{M}$ be two abelian groups such that $\mathfrak{M}$ is a Γ-ring and $e_1 \in \mathfrak{M}$ a nontrivial γ_1-idempotent. Let us consider $e_2 \colon \Gamma \times \mathfrak{M} \to \mathfrak{M}$ and $e_2' \colon \mathfrak{M} \times \Gamma \to \mathfrak{M}$ two $\mathfrak{M}$-additive maps verifying the conditions $e_2(\gamma_1, a) = a - e_1\gamma_1 a$ and $e_2'(a, \gamma_1) = a - a\gamma_1 e_1$. Let us denote $e_2\alpha a = e_2(\alpha, a)$, $a\alpha e_2 = e_2'(a, \alpha)$, $1_1\alpha a = e_1\alpha a + e_2\alpha a$, $a\alpha 1_1 = a\alpha e_1 + a\alpha e_2$ and suppose $(a\alpha e_2)\beta b = a\alpha(e_2\beta b)$ for all $a, b \in \mathfrak{M}$ and $\alpha, \beta \in \Gamma$. Then $1_1\gamma_1 a = a\gamma_1 1_1 = a$ and $(a\alpha 1_1)\beta b = a\alpha(1_1\beta b)$, for all $a, b \in \mathfrak{M}$ and $\alpha, \beta \in \Gamma$, allowing us to write $1_1 = e_1 + e_2$ and $\mathfrak{M}$ as a direct sum of subgroups $\mathfrak{M} = \mathfrak{M}_{11} \oplus \mathfrak{M}_{12} \oplus \mathfrak{M}_{21} \oplus \mathfrak{M}_{22}$, where $\mathfrak{M}_{ij} = e_i\gamma_1\mathfrak{M}\gamma_1 e_j$ $(i, j = 1, 2)$, called *Peirce decomposition* of $\mathfrak{M}$ relative to e_1, satisfying the multiplicative relations:

(i) $\mathfrak{M}_{ij}\Gamma\mathfrak{M}_{kl} \subseteq \mathfrak{M}_{il}$ $(i, j, k, l = 1, 2)$;

(ii) $\mathfrak{M}_{ij}\gamma_1\mathfrak{M}_{kl} = 0$ if $j \neq k$ $(i, j, k, l = 1, 2)$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of a Γ-ring $\mathfrak{M}$ and $\Theta \subseteq \Gamma$, we denote $\mathfrak{A}\Theta\mathfrak{B}$ the subset of $\mathfrak{M}$ consisting of all finite sums of the form $\sum_i a_i\gamma_i b_i$ where $a_i \in \mathfrak{A}$, $\gamma_i \in \Theta$ and $b_i \in \mathfrak{B}$. A *right ideal (resp., left ideal)* of a Γ-ring $\mathfrak{M}$ is an additive subgroup $\mathfrak{I}$ of $\mathfrak{M}$ such that $\mathfrak{I}\Gamma\mathfrak{M} \subseteq \mathfrak{I}$ (resp., $\mathfrak{M}\Gamma\mathfrak{I} \subseteq \mathfrak{I}$). If $\mathfrak{I}$ is both a right and a left ideal of $\mathfrak{M}$, then we say that $\mathfrak{I}$ is an *ideal* or *two-side ideal* of $\mathfrak{M}$.

An ideal $\mathfrak{P}$ of a Γ-ring $\mathfrak{M}$ is called *prime* if for any ideals $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{M}$, $\mathfrak{A}\Gamma\mathfrak{B} \subseteq \mathfrak{P}$ implies that $\mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{B} \subseteq \mathfrak{P}$. A Γ-ring $\mathfrak{M}$ is said to be *prime* if the zero ideal is prime.

Theorem 1.1. *[7, Theorem 4] If $\mathfrak{M}$ is a Γ-ring, the following conditions are equivalent:*

(i) $\mathfrak{M}$ is a prime Γ-ring;

(ii) if $a, b \in \mathfrak{M}$ and $a\Gamma\mathfrak{M}\Gamma b = 0$, then $a = 0$ or $b = 0$.

Let $\mathfrak{M}$ and Γ be abelian groups such that $\mathfrak{M}$ is a Γ-ring. A map $d : \mathfrak{M} \to \mathfrak{M}$ is called a *multiplicative derivation* of $\mathfrak{M}$ if its satisfies the properties $d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$ for all $x, y \in \mathfrak{M}$ and $\gamma \in \Gamma$. We say that a multiplicative derivation d is a *derivation* its is additive, that is, when $d(x+y) = d(x) + d(y)$ for all $x, y \in \mathfrak{M}$.

During the last decade, many mathematicians devoted to study the additivity of maps on associative rings. However, is very difficult to say anything when these applications are defined on arbitrary rings which are not necessarily associative. For the reader interested in applications defined in non-associative rings we recommend some papers [2], [3], [4], [5], [6]. In [1], Daif has asked the following question: When is a multiplicative derivation additive? He answered his question for a multiplicative derivation on a ring having at least one idempotent element. Similarly, we raise the following question: When is a multiplicative derivation of a gamma ring additive? Fortunately, we can give a full answer for this question using similars Martindales conditions when assumed the existence of a family of idempotent elements in $\mathfrak{M}$ which satisfy some conditions.

2 Main theorem

Theorem 2.1. *Let $\mathfrak{M}$ be a Γ-ring containing a family $\{e_\alpha | \alpha \in \Lambda\}$ of γ_α-idempotents which satisfies:*

(i) For each $\alpha \in \Lambda$ there are two $\mathfrak{M}$-additive maps $f_\alpha : \Gamma \times \mathfrak{M} \to \mathfrak{M}$, $f'_\alpha : \mathfrak{M} \times \Gamma \to \mathfrak{M}$ satisfying $f_\alpha(\gamma_\alpha, a) = a - e_\alpha\gamma_\alpha a$, $f'_\alpha(a, \gamma_\alpha) = a - a\gamma_\alpha e_\alpha$, for all $a \in \mathfrak{M}$, such that if we denote $f_\alpha\beta a = f_\alpha(\beta, a)$, $a\beta f_\alpha = f'_\alpha(a, \beta)$, $1_\alpha\beta a = e_\alpha\beta a + f_\alpha\beta a$, $a\beta 1_\alpha = a\beta e_\alpha + a\beta f_\alpha$ (allowing us to write $1_\alpha = e_\alpha + f_\alpha$), then $(a\beta f_\alpha)\gamma b = a\beta(f_\alpha\gamma b)$ for all $a, b \in \mathfrak{M}$ and $\beta, \gamma \in \Gamma$;

(ii) If $x \in \mathfrak{M}$ is such that $x\Gamma\mathfrak{M} = 0$, then $x = 0$;

(iii) If $x \in \mathfrak{M}$ is such that $e_\alpha\Gamma\mathfrak{M}\Gamma x = 0$ for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{M}\Gamma x = 0$ implies $x = 0$);

(iv) For each $\alpha \in \Lambda$ and $x \in \mathfrak{M}$, if $(e_\alpha\gamma_\alpha x\gamma_\alpha e_\alpha)\Gamma\mathfrak{M}\Gamma(1_\alpha - e_\alpha) = 0$ then $e_\alpha\gamma_\alpha x\gamma_\alpha e_\alpha = 0$.

Then any multiplicative derivation d of $\mathfrak{M}$ is additive.

The following lemmas has the same hypotheses of Theorem 2.1 and we need these lemmas for the proof of this theorem. Thus, let us consider $e_1 \in \{e_\alpha | \alpha \in \Lambda\}$ a nontrivial γ_1-idempotent of $\mathfrak{M}$ and $1_1 = e_1 + e_2$.

Lemma 2.1. *(i)* $u\beta\big(d(x+y) - d(x) - d(y)\big) = d\big(u\beta(x+y)\big) - d(u\beta x) - d(u\beta y)$;

(ii) $\big(d(x+y) - d(x) - d(y)\big)\beta u = d\big((x+y)\beta u\big) - d(x\beta u) - d(y\beta u)$, *for all $u, x, y \in \mathfrak{M}$ and $\beta \in \Gamma$.*

Proof. For arbitraries $u, x, y \in \mathfrak{M}$ and $\beta \in \Gamma$ we have $u\beta d(x+y) - u\beta d(x) - u\beta d(y) = d\big(u\beta(x+y)\big) - d(u)\beta(x+y) - d(u\beta x) + d(u)\beta x - d(u\beta y) + d(u)\beta y = d\big(u\beta(x+y)\big) - d(u\beta x) - d(u\beta y)$. Similarly, we prove $\big(d(x+y) - d(x) - d(y)\big)\beta u = d\big((x+y)\beta u\big) - d(x\beta u) - d(y\beta u)$. □

Lemma 2.2. *$d(x_{ii} + x_{jk}) = d(x_{ii}) + d(x_{jk})$ for $j \neq k$.*

Proof. First assume that $i = j$. For arbitraries $a \in \mathfrak{M}$ and $\beta \in \Gamma$ we have $\big(d(x_{ii} + x_{jk}) - d(x_{ii}) - d(x_{jk})\big)\gamma_1 e_l \beta a = d\big((x_{ii} + x_{jk})\gamma_1 e_l \beta a\big) - d(x_{ii}\gamma_1 e_l \beta a) - d(x_{jk}\gamma_1 e_l \beta a) = 0$ $(l = 1, 2)$, since $i \neq k$. Hence $\big(d(x_{ii} + x_{jk}) - d(x_{ii}) - d(x_{jk})\big)\gamma_1 1_1 \beta a = 0$, by condition *(i)* of the Theorem, which implies

$$\big(d(x_{ii} + x_{jk}) - d(x_{ii}) - d(x_{jk})\big)\Gamma\mathfrak{M} = 0. \tag{1}$$

Similarly, in the case of $i \neq j$, we prove that

$$\mathfrak{M}\Gamma\big(d(x_{ii} + x_{jk}) - d(x_{ii}) - d(x_{jk})\big) = 0. \tag{2}$$

From (1) and (2), we conclude that $d(x_{ii} + x_{jk}) = d(x_{ii}) + d(x_{jk})$ for $j \neq k$. □

Lemma 2.3. *$d(x_{12} + y_{12}) = d(x_{12} + y_{12})$.*

Proof. For arbitraries $a \in \mathfrak{M}$ and $\beta \in \Gamma$ by Lemma 2.1, we have $\big(d(x_{12} + y_{12}) - d(x_{12}) - d(y_{12})\big)\gamma_1 e_1 \beta a = d\big((x_{12} + y_{12})\gamma_1 e_1 \beta a\big) - d(x_{12}\gamma_1 e_1 \beta a) - d(y_{12}\gamma_1 e_1 \beta a) = 0$. It follows that

$$\big(d(x_{12} + y_{12}) - d(x_{12}) - d(y_{12})\big)\gamma_1 e_1 \beta a = 0. \tag{3}$$

Next, we have $\big(d(x_{12} + y_{12}) - d(x_{12}) - d(y_{12})\big)\gamma_1 e_2 \beta a = d\big((x_{12} + y_{12})\gamma_1 e_2 \beta a\big) - d(x_{12}\gamma_1 e_2 \beta a) - d(y_{12}\gamma_1 e_2 \beta a) = d\big(x_{12}\gamma_1(e_2 \beta a + y_{12}\gamma_1 e_2 \beta a) + e_1\gamma_1(e_2 \beta a + y_{12}\gamma_1 e_2 \beta a)\big) - d\big(x_{12}\gamma_1(e_2 \beta a + y_{12}\gamma_1 e_2 \beta a)\big) - d\big(e_1\gamma_1(e_2 \beta a + y_{12}\gamma_1 e_2 \beta a)\big) = \big(d(x_{12} + e_1) - d(x_{12}) - d(e_1)\big)\gamma_1(e_2 \beta a + y_{12}\gamma_1 e_2 \beta a) = 0$, by Lemma 2.2, which implies

$$\big(d(x_{12} + y_{12}) - d(x_{12}) - d(y_{12})\big)\gamma_1 e_2 \beta a = 0. \tag{4}$$

From (3) and (4) we have $\big(d(x_{12}+y_{12}) - d(x_{12}) - d(y_{12})\big)\gamma_1 1_1 \beta a = 0$, by condition *(i)* of the Theorem, which implies

$$\big(d(x_{12} + y_{12}) - d(x_{12}) - d(y_{12})\big)\Gamma\mathfrak{M} = 0.$$

By condition *(ii)* of the Theorem, we can conclude that $d(x_{12} + y_{12}) = d(x_{12}) + d(y_{12})$. □

Lemma 2.4. *$d(x_{11} + y_{11}) = d(x_{11}) + d(y_{11})$.*

Proof. First of all, note that $d(x_{11}+y_{11}) - d(x_{11}) - d(y_{11}) = e_1\gamma_1\big(d(x_{11}+y_{11}) - d(x_{11}) - d(y_{11})\big)\gamma_1 e_1$, by Lemma 2.1. Hence, for arbitraries $a \in \mathfrak{M}$ and $\alpha, \beta \in \Gamma$ we have $\big(d(x_{11} + y_{11}) - d(x_{11}) - d(y_{11})\big)\gamma_1 e_l \alpha a \beta e_2 = d\big((x_{11} + y_{11})\gamma_1 e_l \alpha a \beta e_2\big) -$

$d(x_{11}\gamma_1 e_l\alpha a\beta e_2) - d(y_{11}\gamma_1 e_l\alpha a\beta e_2) = 0$ $(l = 1, 2)$, by Lemma 2.3. It follows that $\big(d(x_{11}+y_{11}) - (d(x_{11}) + d(y_{11})\big)\gamma_1 1_1\alpha a\beta e_2 = 0$ which implies

$$\Big(e_1\gamma_1\big(d(x_{11}+y_{11}) - d(x_{11}) - d(y_{11})\big)\gamma_1 e_1\Big)\Gamma\mathfrak{M}\Gamma(1_1 - e_1) = 0.$$

Thus, $d(x_{11}+y_{11}) = d(x_{11}) + d(y_{11})$, by condition (iv) of the Theorem. □

Lemma 2.5. *d is additive in $\mathfrak{M}_{11} \oplus \mathfrak{M}_{12}$.*

Proof. For arbitraries $a_{rs} \in \mathfrak{M}_{rs}$ $(r, s = 1, 2)$ and $\beta \in \Gamma$, by Lemmas 2.1, 2.3 and 2.4, we have $\big(d(x_{11}+x_{12}+y_{11}+y_{12}) - d(x_{11}+x_{12}) - d(y_{11}+y_{12})\big)\gamma_1 e_l\beta a_{rs} = d\big((x_{11}+x_{12}+y_{11}+y_{12})\gamma_1 e_l\beta a_{rs}\big) - d\big((x_{11}+x_{12})\gamma_1 e_l\beta a_{rs}\big) - d\big((y_{11}+y_{12})\gamma_1 e_l\beta a_{rs}\big)$ $= 0$ $(l = 1, 2)$ resulting in $\big(d(x_{11}+x_{12}+y_{11}+y_{12}) - d(x_{11}+x_{12}) - d(y_{11}+y_{12})\big)\gamma_1 e_l\beta a = 0$ $(l = 1, 2)$, where $a = a_{11} + a_{12} + a_{21} + a_{22}$, by condition *(i)* of the Theorem. It follows that $\big(d(x_{11}+x_{12}+y_{11}+y_{12}) - d(x_{11}+x_{12}) - d(y_{11}+y_{12})\big)\gamma_1 1_1\beta a = 0$ which implies

$$\big(d(x_{11}+x_{12}+y_{11}+y_{12}) - d(x_{11}+x_{12}) - d(y_{11}+y_{12})\big)\Gamma\mathfrak{M} = 0.$$

It follows that, $d(x_{11}+x_{12}+y_{11}+y_{12}) = d(x_{11}+x_{12}) + d(y_{11}+y_{12})$, by condition *(ii)* of the Theorem. □

Lemma 2.6. *d is additive in $e_1\Gamma\mathfrak{M}$.*

Proof. For arbitraries $x, y \in \mathfrak{M}$ and $\alpha, \beta \in \Gamma$ we have $e_1\alpha x, e_1\beta y \in \mathfrak{M}_{11} \oplus \mathfrak{M}_{12}$. Hence, $d(e_1\alpha x + e_1\beta y) = d(e_1\alpha x) + d(e_1\beta y)$, by Lemma 2.5. □

Proof of Theorem 2.1.

Proof. Let $x, y \in \mathfrak{M}$. Since d is additive on $e_\alpha\Gamma\mathfrak{M}$ for all $\alpha \in \Lambda$, by Lemma 2.6, then for an arbitrary element $r \in \mathfrak{M}$ and elements $\lambda, \mu \in \Gamma$ we have

$$\begin{aligned} & e_\alpha\lambda r\mu\big(d(x+y) - d(x) - d(y)\big) \\ = \; & d\big(e_\alpha\lambda r\mu(x+y)\big) - d(e_\alpha\lambda r\mu x) - d(e_\alpha\lambda r\mu y) \\ = \; & d(e_\alpha\lambda r\mu x + e_\alpha\lambda r\mu y) - d(e_\alpha\lambda r\mu x) - d(e_\alpha\lambda r\mu y) \\ = \; & 0. \end{aligned}$$

It follows that

$$e_\alpha\Gamma\mathfrak{M}\Gamma\big(d(x+y) - d(x) - d(y)\big) = 0$$

for all $\alpha \in \Lambda$. From condition *(iii)* of the Theorem, we conclude that $d(x+y) = d(x) + d(y)\big)$. This shows that d is additive and therefore a derivation on $\mathfrak{M}$. □

Corollary 2.1. *Let $\mathfrak{M}$ be a prime Γ-ring containing a γ_1-idempotent e_1 ($\mathfrak{M}$ need not have a γ_1-identity element), where $\gamma_1 \in \Gamma$. Suppose $e_2 : \Gamma \times \mathfrak{M} \to \mathfrak{M}$, $e_2' : \mathfrak{M} \times \Gamma \to \mathfrak{M}$ two $\mathfrak{M}$-additive maps such that $e_2(\gamma_1, a) = a - e_1\gamma_1 a$, $e_2'(a, \gamma_1) = a - a\gamma_1 e_1$, for all $a \in \mathfrak{M}$, and if we denote $e_2\alpha a = e_2(\alpha, a)$, $a\alpha e_2 = e_2'(a, \alpha)$, $1_1\alpha a = e_1\alpha a + e_2\alpha a$, $a\alpha 1_1 = a\alpha e_1 + a\alpha e_2$, then $(a\alpha e_2)\beta b = a\alpha(e_2\beta b)$ for all $\alpha, \beta \in \Gamma$ and $a, b \in \mathfrak{M}$. Then any multiplicative derivation d of $\mathfrak{M}$ is a derivation.*

Proof. The result follows directly from the Theorem 2.1. □

Corollary 2.2. *Let $\mathfrak{M}$ be a prime Γ-ring containing a γ_1-idempotent and a γ_1-unity element, where $\gamma_1 \in \Gamma$. Then any multiplicative derivation d of $\mathfrak{M}$ is a derivation.*

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MATHEMATICAL THEORY OF SPACE-TIME WITH SASC

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Abstract

Using space-time ring we establish the mathematical theory of space-time with subluminal and superluminal coexistence (SASC). Using two methods we deduce the new gravitational formula. Tardyonic rotating motion produces the centrifugal force, but tachyonic rotating motion produces the centripetal force, that is gravity. Using it we establish the expansion theory of the universe and suggest the new universe model. We prove that in the universe there are no dark matter and no dark energy. New gravitational formula changes all that. Multiverse and gravitational waves do not exist. Theory of everything has the two forces: (1) the subluminal force (electromagnetism and weak force) and (2) superluminal force (gravity and strong force). We prove that equivalence principle is not exist. Therefore we prove that general relativity is the biggest wrong theory.

Keywords: new gravitational formula, expansion theory of universe.
PACS:04.90.+e,98.80.-k.

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1. Mathematical Theory of Space-Time With SASC

Throughout history, the notion of space and time has undergone a number of dramatic transformations, thanks to figures ranging from Aristotle, Leibniz and Newton to Gauss, Poincare and Einstein. In the present understanding of nature, space and time form a single entity called space-time. In 1975 Jiang has established the mathematical theory of space-time with SASC. This theory plays a key role for the entire field of physics to unlock the deepest mysteries of the universe.

In the Universe there are two matters: (1) observable subluminal matter called tardyon(locality) and (2) unobservable superluminal matter called tachyon(non-locality) which coexist in motion. Tachyon can be converted into tardyon, and *vice versa*. Tardyonic rotating motion produces the centrifugal force, but tachyonic rotating motion produces the centripetal force, that is gravity. Using tardyonic and tachyonic coexistence principle we deduce the new gravitational formula,

For establishing the mathematical theory of space-time with SASC. we first define space-time ring [1,7-8]

$$Z = \begin{pmatrix} ct & x \\ x & ct \end{pmatrix} = ct + jx, \tag{1}$$

where x and t are the tardyonic space and time coordinates, c is light velocity in vacuum, $j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(1) can be written as Euler form

$$Z = ct_0 e^{j\theta} = ct_0(\mathrm{ch}\,\theta + j\,\mathrm{sh}\,\theta), \tag{2}$$

where ct_0 is the tardyonic invariance, θ tardyonic hyperbolical angle.

From (1) and (2) we have

$$ct = ct_0 \,\mathrm{ch}\,\theta, \quad x = ct_0 \,\mathrm{sh}\,\theta \tag{3}$$

$$ct_0 = \sqrt{(ct)^2 - x^2}. \tag{4}$$

From (3) we have

$$\theta = \mathrm{th}^{-1}\frac{x}{ct} = \mathrm{th}^{-1}\frac{u}{c}. \tag{5}$$

where $c \geq u$ is the tardyonic velocity.

Using the morphism $j : z \to jz$, we have

$$jz = \bar{x} + jc\bar{t} = \bar{x}_0 e^{j\bar{\theta}} = \bar{x}_0(\mathrm{ch}\,\bar{\theta} + j\,\mathrm{sh}\,\bar{\theta}), \tag{6}$$

where $\bar{x}$ and $\bar{t}$ are the tachyonic space and time coordinates, $\bar{x}_0$ is tachyonic invariance, $\bar{\theta}$ tachyonic hyperbolical angle.

From (6) we have

$$\bar{x} = \bar{x}_0 \operatorname{ch}\bar{\theta}, \quad c\bar{t} = \bar{x}_0 \operatorname{sh}\bar{\theta}. \tag{7}$$

$$\bar{x}_0 = \sqrt{(\bar{x})^2 - (c\bar{t})^2}\,. \tag{8}$$

From (7) we have

$$\bar{\theta} = \operatorname{th}^{-1}\frac{c\bar{t}}{\bar{x}} = \operatorname{th}^{-1}\frac{c}{\bar{u}}. \tag{9}$$

where $\bar{u} \geq c$ is the tachyonic velocity.

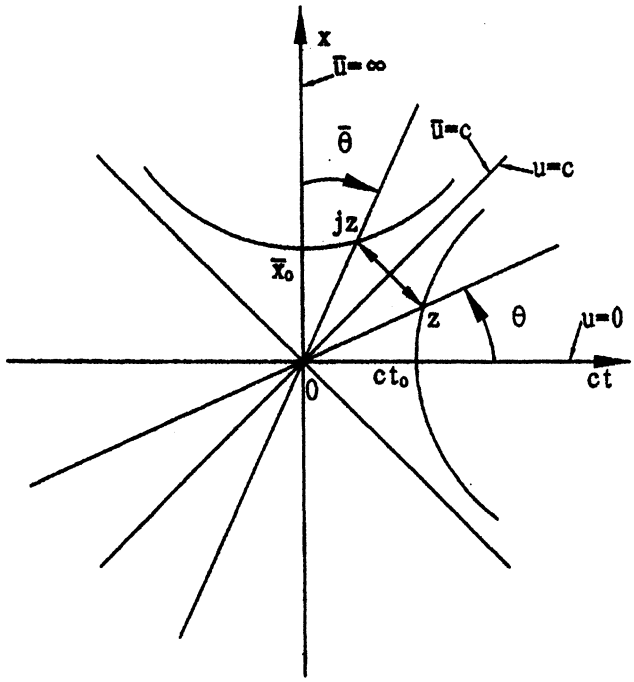

Fig. 1. Mathematical theory of space-time with SASC

Figure 1 shows the formulas (1)-(9). $j : z \to jz$ is that tardyon can be converted into tachyon, but $j : jz \to z$ is that tachyon can be converted into tardyon. $u = 0 \to u = c$ is the positive acceleration, but $\bar{u} = \infty \to \bar{u} = c$ is the negative acceleration, which coexist. At the ct-axis u=0 and x=0 we define the tardyonic rest time t. At the $x-$axis we define the tachyonic rest space

$$\bar{X}_0 = \lim_{\substack{\bar{u}\to\infty \\ t\to 0}} \bar{u}t = \text{constant}. \tag{10}$$

Since at rest the tachyonic rest time $t = 0$ and $\bar{u} = \infty$, we prove that tachyon is unobservable. Fig.1 and (10) are mathematical theory of space-time with SASC, which are the foundations of physics and cosmology. Using it we prove that dark matter, dark energy, gravitational waves, quantum entanglements, quantum computers and black holes do not exist. From Fig.1 we deduce new gravitational formula.

2. New Gravitational Formula: $\overline{F} = -\dfrac{mc^2}{R}$ without dark matter

Assume $\theta = \overline{\theta}$, from (5) and (9) we get the tardyonic and tachyonic coexistence principle [1-4,7-8]

$$u\overline{u} = c^2. \tag{11}$$

Using the analytical method, we deduce the new gravitational formula. Differentiating (11) by the time, we get

$$\frac{d\overline{u}}{dt} = -\left(\frac{c}{u}\right)^2 \frac{du}{dt}. \tag{12}$$

$\dfrac{du}{dt}$ and $\dfrac{d\overline{u}}{dt}$ can coexist in motion, but their directions are opposite.

We study the tardyonic and tachyonic rotating motions. In 1673 Huygens discovered that the tardyonic rotation produces centripetal acceleration

$$\frac{du}{dt} = \frac{u^2}{R}, \tag{13}$$

where R is rotating radius.

Substituting (13) into (12) we have the tachyonic centrifugal acceleration

$$\frac{d\overline{u}}{dt} = -\frac{c^2}{R}. \tag{14}$$

(13) and (14) have the same form. From (13) we get the tardyonic centrifugal force

$$F = \frac{Mu^2}{R}, \tag{15}$$

where M is the inertial mass.

From (14) we get the tachyonic centripetal force, that is gravity

$$\overline{F} = -\frac{mc^2}{R}, \tag{16}$$

where m is the gravitational mass converted into by tachyonic mass $\overline{m}$.

(15) and (16) have the same form. (16) is the new gravitational formula.

Using the geometrical method we deduce the new gravitational formula.

Figure 2 shows that the rotation ω of body A emits tachyon mass $\overline{m}$, which forms the

tachyon and gravitation field and gives the body B revolutions u and $\bar{u}$.

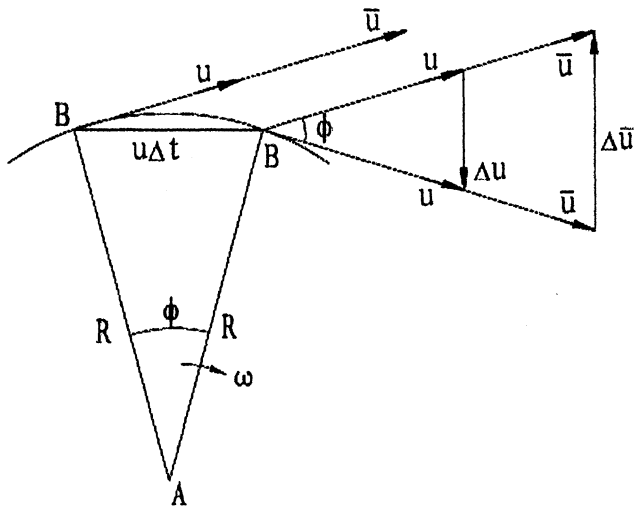

Fig.2. On body B the $\frac{du}{dt}$ and $\frac{d\bar{u}}{dt}$ coexistence [2]

From Fig. 2 .it follows

$$\frac{u\Delta t}{R}=\frac{\Delta u}{u}. \tag{17}$$

From (17) it follows the tardyon centripetal acceleration on the body B [2-4],

$$\frac{du}{dt}=\lim_{\substack{\Delta u\to 0\\ \Delta t\to 0}}\frac{\Delta u}{\Delta t}=\frac{u^2}{R}. \tag{18}$$

From Fig. 2. it follows

$$\frac{u\Delta t}{R}=-\frac{\Delta\bar{u}}{\bar{u}}. \tag{19}$$

From (19) and (11) it follows the tachyon centrifugal acceleration on the body B [2-4],

$$\frac{d\bar{u}}{dt}=\lim_{\substack{\Delta\bar{u}\to 0\\ \Delta t\to 0}}\frac{\Delta\bar{u}}{\Delta t}=-\frac{u\bar{u}}{R}=-\frac{c^2}{R}. \tag{20}$$

On body B the $\frac{du}{dt}$ and $\frac{d\bar{u}}{dt}$ coexistence.

From (18) it follows the tardyon centrifugal force on body B [2-4],

$$F=\frac{M_B u^2}{R}, \tag{21}$$

where M_B is body B inertial mass.

From (20) it follows the tachyon centripetal force on body B, that is gravity [2-4,7-8],

$$\overline{F} = -\frac{mc^2}{R}, \qquad (22)$$

where m is the gravitation mass converted into by tachyon mass $\overline{m}$ which is unobservable, but m is observable.

(22) is the new gravitational formula. In 1976[2] this simple thought made a deep impression on me. It impelled me to establish the new gravitational theory. On body B the F and $\overline{F}$ coexistence.

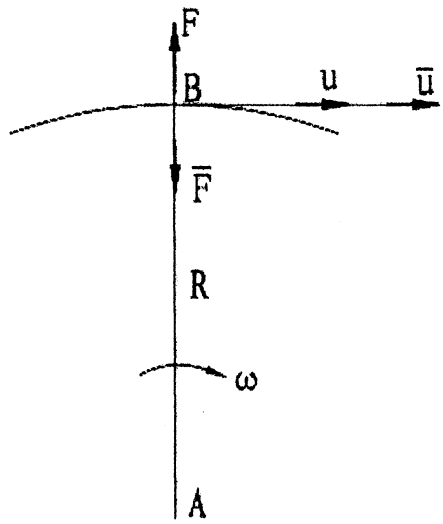

Fig.3. On body B the F and $\overline{F}$ coexistence and the new Universe model

From Fig. 3, it follows

$$F + \overline{F} = 0. \qquad (23)$$

From (21), (22) and (23) it follows

$$\frac{m}{M_B} = \frac{u^2}{c^2}. \qquad (24)$$

Body B increases mass m and centrifugal force is greater than gravitation force, then body B expands outward. Dark matter which causes cosmic attraction is wrong.

From above we prove that equivalence principle is not exist. Therefore, we prove that general relativity is the biggest wrong theory.

Using Fig.3 we establish the new universe model.

(1) Body A is earth center, body B is the moon;

(2) Body A is the sun center, body B is the Earth;

(3) Body A is Galactic center, body B is stars in the milky way;

(4) Body A is the universe center; body B is the stars in the universe.

In Fig.3 on body B there is tardyonic centrifugal force and tachyonic centripetal force coexistence. It is the universe foundations. All matter has unobservable tachyons to produce gravity. We should establish new cosmological theory and new particle physics theory.

From (22) it follows Newtonian gravitation formula. The m is proportional to body A mass M_A, in (24) m is proportional to M_B, is inversely proportional to the distance R between body A and body B. It follows

$$m = k\frac{M_A M_B}{R}, \tag{25}$$

where k is constant

Substituting (25) into (22) it follows the Newtonian gravitation formula [2-4,7-8]

$$\overline{F} = -G\frac{M_A M_B}{R^2}, \tag{26}$$

where $G = kc^2 = 6.673\times10^{-8}\ \mathrm{cm^3/g\cdot sec^2}$ is gravitation constant.

Now we study the freely falling body. Tachyonic mass $\overline{m}$ can be converted into gravitational mass m, which acts on the freely falling body and produces the gravitational force

$$\overline{F} = -\frac{mc^2}{R}, \tag{27}$$

where R is the Earth radius.

We have the equation of motion

$$\frac{mc^2}{R} = Mg, \tag{28}$$

where g is gravitational acceleration, M is inertial mass of freely falling body.

From (28) it follows the gravitational coefficient

$$\eta = \frac{m}{M} = \frac{Rg}{c^2} = 6.9\times10^{-10}. \tag{29}$$

Eötvös experiment $\eta \sim 5\cdot10^{-9}$ and Dicke experiment $\eta \sim 10^{-11}$. Since the gravitational mass m can be transformed into the rest mass in freely falling body, we prove that the freely falling bodies fall with the same acceleration and the equivalence principle is not exist. Therefore, we prove that general relativity is the biggest wrong theory.

3. The expansion theory of the universe without dark energy

Using new gravitational formula, we study the expansion theory of the Universe [7-8]. Figure 4 shows an expansion model of the Universe. The rotation ω_1 of body A emits tachyonic flow, which forms the tachyonic field. Tachyonic mass $\overline{m}$ acts on body B, which produces its rotation ω_2, revolution u and gravitational force

$$\overline{F}_1 = -\frac{mc^2}{R}, \tag{30}$$

where R denotes the distance between body A and body B, m is gravitational mass converted into by tachyonic mass $\overline{m}$ which is unobservable but m is observable.
The rotation of the body B around body A produces the centrifugal force

$$F_1 = \frac{M_B u^2}{R}, \tag{31}$$

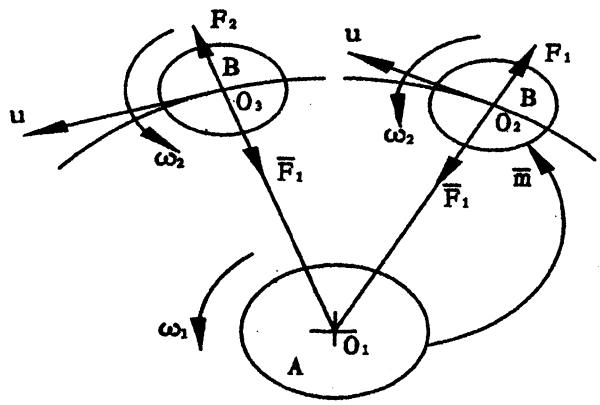

Fig. 4. An expansion model of the Universe

where M_B is the inertial mass of body B, u is the orbital velocity of body B.
At the O_2 point we assume

$$F_1 + \overline{F}_1 = 0. \tag{32}$$

From (32) it follows that the coexistence of the gravitational force and centrifugal force.
From (30)-(32) it follows the gravitational coefficient

$$\eta = \frac{m}{M_B} = \left(\frac{u}{c}\right)^2. \tag{33}$$

At the O_3 point the tachyonic mass $\overline{m}$ can be converted into the rest mass m in body B, it follows

$$F_2 = \frac{M_B u^2}{R} + \frac{mu^2}{R}. \qquad (34)$$

Since $F_2 + \overline{F}_1 > 0$, centrifugal force F_2 is greater than gravitational force $\overline{F}_1$, then the body B expands outwards and its mass increases. This is a expansion mechanism of the Universe. From (31,32,34) we have

$$F_2 + \overline{F}_1 = \frac{mu^2}{R} = M_B g_e \qquad (35)$$

From (35) we obtain the expansion acceleration

$$g_e = mu^2 / M_B R \qquad (36)$$

Substituting (33) in (36) we obtain

$$g_e = \frac{u^4}{c^2 R} \qquad (37)$$

If body A is the Earth, then body B is the Moon; if body A is the Sun, then body B is the Earth; ···. It can explain our accelerating universe. In the universe there are no dark matter and no dark energy. This simple thought made a deep impression on me. It impelled me to establish a expansion theory of the universe. Dark energy responsible cosmic repulsion is wrong.

If the body A is the Sun and body B is the planet. We calculate the gravitational coefficients η as shown in table 1.

Table 1: Values of the gravitational coefficients η

Planet	u (km/sec)	$\eta(10^{-10})$
Mercury	47.89	255.2
Venus	35.03	136.5
Earth	29.79	98.7
Mars	24.13	64.8
Jupiter	13.06	19.0
Saturn	9.64	10.3
Uranus	6.81	5.2
Neptune	5.43	3.3
Pluto	4.74	2.5

The gravitational field of the solar system is the origin of the planet mass. From it the planet acquire mass. In the universe there are no Higgs particles. The equivalence principle is not exist. Therefore, we prove that general relativity is the biggest wrong theory.

4. Conclusion

In summary. We deduce tardyonic and tachyonic coexistence principle. Using it we deduce the centrifugal formula and new gravitational formula We establish the expansion theory of the universe without dark energy. The new gravitational formula is foundations of particle physics and cosmology. We prove that in the universe no dark matter, no dark energy, no gravitational waves and no quantum gravity.

Where did we come from? Where are we going? What makes up the universe? These questions have occupied mankind for thousands of years. Over the course of history, our view of the world has been changed. Theologians and philosophers, physicists and astronomers have given us very different answers. Where did we come from? We answer this questions this way $\overline{m} \to m$, tachyons $\to$ tardyons[1], that is tachyons can be converted into the electrons and positrons which are the basic building-blocks of the elementary particles[5-6]. In the universe there are no Higgs particles and no quarks. The standard model of particle physics is wrong. The tachyons are the origin of mass. Where are we going? We answer this question this way $m \to \overline{m}$, that is the tardyons produce tachyons [1]. The tardyons and tachyons make up the Universe.

Note: In 1976 Jiang found a gravitational formula [2]: $\overline{F} = -\overline{m}c^2/R$, where $\overline{m}$ is the tachyonic mass. In 2004 Jiang studied the Universe expansion and found a new gravitational formula $\overline{F} = -mc^2/R$, where m is gravitational mass converted into by tachyonic mass $\overline{m}$ [7].

Compatibility of arbitrary speeds for interior dynamical problems implies a new scientific era [9] Newtonian gravity formula is based on empirical evidence. He did not explain what is gravity? how it works? Equivalence principle does not exist, the gravitation must not be described by a "metric theory". The Einstein field equations are wrong. therefore, we prove that general relativity is the biggest wrong theory.

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THERE ARE INFINITELY MANY PRIME TRIPLETS P,3P-2, 3P+2

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Abstract

Using Jiang's function, we prove that there are infinitely many primes P such that $3P-2$ and $3P+2$ are primes.

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In studying Williams numbers Echi conjectures that there are infinitely many prime triplets $P, 3P-2, 3P+2$ [1]. In this paper using Jiang's function we prove this conjecture and find the best asymptotic formula for the number of primes P.

Theorem 1. The prime equations are

$$P_1 = 3P-2 \text{ and } P_2 = 3P+2 \quad (1)$$

There are infinitely many primes P such that P_1 and P_2 are primes.

Proof.

Firstly, Jiang's function is

$$J_2(\omega) = \prod_P (P-1-x(P)) \quad (2)$$

where $\omega = \prod_P P$, $x(P)$ is the number of solutions of the following congruence

$$(3q-2)(3q+2) = 0 (\text{mod } P) \quad (3)$$

where q=1,2,...p-1

From (3) we obtain

Table 1

q	$3q-2$	$3q+2$	
1	1	5	$x(2)=0, J_2(2)=1$
2	4	8	$x(3)=0, J_2(3)=2$
3	7	11	
4	2×5	2×7	$x(5)=2, J_2(5)=2$
5	13	17	
6	16	20	$x(7)=2, J_2(7)=4$
7	19	23	
8	2×11	2×13	
9	25	29	
10	28	32	$x(11)=2, J_2(11)=8$
...	...	...	...
$P-1$	$3P-5$	$3P-1$	$x(P)=2, J_2(P)=P-3$

In order to understand the Jiang's function, we explain the table 1.

Let $\omega = 2$, Euler function $\phi(2) = 1$. There is the prime equation

$$2n+1 \qquad (4)$$

Where n=1,2,....

$J_2(2) = 1$, there is the prime equation

$$P = 2n+1 \qquad (5)$$

Substituting (5) into (1) we obtain

$$P_1 = 6n+1 \text{ and } P_2 = 6n+5 \qquad (6)$$

There are infinitely many integers n such that P, P_1 and P_2 are primes.

Let $\omega = 6$, $\phi(6) = 2$. There are the prime equations

$$6n+h \qquad (7)$$

Where n=0.1.2,....h=1,5..

$J_2(6) = 2$, there are the prime equations

$$P = 6n+h \qquad (8)$$

Substituting (8) into (1) we obtain

$$P_1 = 18n+3h-2 \text{ and } P_2 = 18n+3h+2. \qquad (9)$$

There are infinitely many integers n such that P, P_1 and P_2 are primes.

Let $\omega = 30$, $\phi(30) = 8$. There are the prime equations

$$30n+h \qquad (10)$$

Where h=1,7,11,13,17,19,23,29, $n = 0,1,...$

$J_2(30) = 4$, there are the prime equations

$$P = 30n+u, \qquad (11)$$

where $u = 7,13,17,23$.

Substituting (11) into (1)

$$P_1 = 90n+3u-2 \text{ and } P_2 = 90n+3u+2 \qquad (12)$$

There are infinitely many integers n such that P, P_1 and P_2 are primes.

Let $\omega = 210$, $\phi(210) = 48$. There are the prime equations

$$210n+h \qquad (13)$$

where h=1, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 121, 127, 131, 137, 139, 143, 149, 151, 157, 163, 167, 169, 173, 179, 181, 187, 191, 193, 197, 199, 209, $n = 0,1,2...$

$J_2(210)=16$, there are the prime equations

$$P=210n+u \tag{14}$$

where u=13, 23, 37, 43, 47, 83, 97, 103, 107, 113, 127, 163, 167, 173, 187, 197.

Substituting (14) into (1) we obtain

$$P_1=630n+3u-2 \quad \text{and} \quad P_2=630n+3u+2 \tag{15}$$

There are infinitely many integers n such that P, P_1 and P_2 are primes.

Let $\omega=2310$, $\phi(2310)=480$. There are the prime equations

$$2310n+h, \tag{16}$$

Where n=0,1,....,h=13,17,....2309..

$J_2(2310)=128$, there are the prime equations

$$P=2310n+u, \tag{17}$$

where u=13, 17,... 2287, 2297.

Substituting (17) into (1)

$$P_1=6930n+3u-2 \quad \text{and} \quad P_2=6930n+3u+2. \tag{18}$$

There are infinitely many integers n such that P, P_1 and P_3 are primes.

$J_2(\omega)\to\infty$ as $\omega\to\infty$, there infinitely many prime equations such that P, P_1 and P_2 are primes.

We prove the theorem 1

From (2) we obtain

$$J_2(\omega)=2\prod_{5\le P}(P-3) \tag{19}$$

Secondly, we obtain the best asymptotic formula for the number of primes P [2]

$$\pi_3(N,2)=\left|\{P\le N, 3P-2=prime, 3P+2=prime\}\right|\sim\frac{J_2(\omega)\omega^2}{\phi^3(\omega)}\frac{N}{\log^3 N}$$

$$=9\prod_{5\le P}\left(1-\frac{3P-1}{(P-1)^3}\right)\frac{N}{\log^3 N}\sim 5.77\frac{N}{\log^3 N} \tag{20}$$

where $\phi(\omega)=\prod_P(P-1)$.

From (20) we obtain

Table 2

	10^2	10^3	10^4	10^5	10^6	10^7	10^8
$\pi_3(N,2)$ (20)	6	18	74	378	2188	13780	92312
$\pi_3(N,2)$ (exact)	7	21	89	445	2420	14828	98220

From table 2 we consider that prime distribution is order rather than random [2].
Using the circle method and the sieve methods one cannot prove any prime problems. From Hardy-Littlewood(1923) to now all prime papers are conjecture and wrong.. Using Jiang function, we prove about all prime problems in prime distribution.

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SIMPLE MATHEMATICAL JOINT RELATION AND ITS APPLICATIONS IN INTERSECTING SCIENCES, BIOLOGY AND SOCIAL SCIENCE, ETC.

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Abstract

For two any sets M and N, we research the simple mathematical joint relation $(M,k)(k',N) = (M,kk',N)$ of algebra in set theory, and discuss its some properties. It may describe various relations among sciences and many intersecting sciences. This is applied to various aspects in natural science and social science, which includes transcription, splicing and replication, etc., in biology, and both main relations in micro-economics and macro-economics, etc. Further, it may be corrected and developed.

Key words: algebra, joint relation, set, mathematical application, intersecting science, biology, economics, social science.

Mathematics Subject Classification (2010): 03E20; 62P10; 92B05; 62P20; 91B02; 62P25

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1. Introduction

The nonstandard analysis is a great development in calculus [1]. Polya and Szego discussed generally problems and theorems in analysis [2].

From 1978 Santilli discovered the new Iso-Mathematics, which includes isonumbers, isorepresentation [3-5], etc., and they are applied for many regions [5,6].

Based on a brief review on developments of number system, we proposed a new developed pattern. The quaternion is extended to a matrix form $aI+bC+cB+dA$, in which the unit matrix I and three special matrices C,B,A correspond to number 1 and three units of imaginary number i,j,k, respectively. They form usually a ring. But some fields may be composed of some special 2-rank, even n-rank matrices, for example, three matrices $aI+bC$, $aI+cB$, $aI+dA$ and so on. It is a new type of hypercomplex number fields. Further, the physical applications and possible meaning of the new number system are researched [7]. Moreover, we researched graph theory and its developments [8,9], and discussed the topological physics, the general topological sciences [10], and the general uncertain sciences [11].

In general nonlinear mathematics, we researched the nonlinear geometry, analyze and algebra. Then the general nonlinear methods and various nonlinear physics are discussed. Further, we searched the general nonlinear sciences, in instance, nonlinear astronomy, biology, Earth science, and social sciences, etc. In particular, we researched chaos and fractal and their applications [12], and general nonlinear quantum theory [13,14]. We proved that some nonlinear equations have double solutions of soliton and chaos, in which all nonlinear equations with a soliton solution may derive chaos, but only some equations with a chaos solution have a soliton. The conditions of the two solutions are different. When some parameters are certain constants, the soliton is derived; while these parameters vary in a certain region, the bifurcation-chaos appears. It connects a chaotic control probably. We discussed some possible meanings on the double solutions in mathematics, physics, particle theory and neurobiology [15]. In this paper, we research a simple mathematical joint relation, and discuss its applications in intersecting sciences, biology and social sciences, etc.

2. Simple Mathematical Joint Relation

For two any sets M and N, we research a simple mathematical joint relation of

algebra in relations of set theory:

$$(M,k)(k',N) = M,kk',N). \quad (1)$$

Here k and k' may be the same or different operators or quantities, for example, number, scale, vector, tensor, matrix, etc. And $kk' = j$ is a joint relation between M and N, which is but only those known union, intersection, difference, and so on. This may be addition, product, etc., and may indeed be part addition and product, i.e., an incomplete joint. It is a type of conglutination or adhesion joints. Complete joint for k and k' with the same types is called the fastener joint.

General addition of matrices must be both of the same shape, but it may be:

$$\begin{pmatrix}2 & 8\\ 3 & 5\end{pmatrix}+\begin{pmatrix}3 & 6 & 1\\ 4 & 7 & 9\end{pmatrix}=\begin{pmatrix}5 & 14 & 1\\ 7 & 12 & 9\end{pmatrix}, \text{ or } \begin{pmatrix}5 & 6 & 9\\ 7 & 7 & 14\end{pmatrix}, \text{ or } \begin{pmatrix}3 & 8 & 9\\ 4 & 10 & 14\end{pmatrix}. \quad (2)$$

Result is not alone, which is determined by both different relations. Product is more complex. This mathematical basis is probably related with K.Godel's incompleteness theorem, and multi-choice of Zermelo-Fraenkel-Church (ZFC) axiom system in set theory.

If k and k' are different types, for example, k is number and k' is matrix, vector, etc., so kk'=j is scalar product.

If kk'=j=k'k, joint is symmetric. Contrarily, kk'=j≠k'k is asymmetric.

The joint relation is related with graph, and is possibly an algebraic form of related set in graph theory. It may possess different strengths, for example, k=3,k'=5>k=4,k'=2 for product; and possess interacting distances by matrix, etc. Both correspond to directed graph and weighted graph.

A particular example is complete fusion and unification, i.e., (M,k)(k',N)= (M,N), here $kk' = j = 1$.

Inverse operation is decomposition: $j \to kk'$.

Further, the unidirectional joint develops to the two-directional, i.e.,

$$(M,k_1)(k_2,N,k_3 k_4,L,\ldots) = M,j_{12},N,j_{34},L,\ldots. \quad (3)$$

It is also related with composition relation and transitivity.

We may apply different expressions for attraction by k)(k' and by M(k, and for repulsion by k)N. Moreover, k(c,M) represses k acts at position c in M. Thom proposed mathematical models of morphogenesis applied widely [16]. This tool is a relation between border fields.

From this we express that a set attracts m by $\sum_i m_i(k_i,S)$, and repulses m' by $(S',k'_j)\sum_j m'_j$. In ecology both may express the suitable conditions and unsuitable conditions for the same species. If both interacting strengths are the same k=k', S>S' or S<S' may express comparison on markets of both commodities, and on social forces of different religions.

This simple mathematical tool may be applied to various aspects in natural science and social science. First, it describes simply various relations among different sciences and intersecting sciences, for example, physics and chemistry by (P,m)(a,C), here m is molecule and a is atom; quantum biology by B(q,P); statistical economics by E(s,M), here M is mathematics; mathematical-physical equations by M(e,P); molecular biology by $B(m,\overset{P}{C})$; physical chemistry is $C(\overset{m}{a},P)$, and so on.

3. Applications in Physics and Chemistry

For physics, it may include that gravitational interactions exchange gravitons:

$$(m,\frac{k}{r})(-\frac{1}{r},M) = -\frac{kmM}{r^2}. \tag{4}$$

Electromagnetic interactions exchange photons:

$$(e,\frac{1}{r})(\pm\frac{1}{r},Q) = \pm\frac{eQ}{r^2}. \tag{5}$$

Strong interactions exchange mesons and gluons, and weak interactions exchange $(W^{\pm},Z^0)$. For early universal four fermions interactions as $n \to p + e^- + \bar{\nu}$, i.e., $(n,e^-)(p,\bar{\nu})$, etc. Nuclear fission is $\mathrm{n(k,N)} = (\mathrm{N}_1,k)N_2$, and nuclear fusion is

$(N_1,k)(e,N_2)=N$.

For chemistry, chemical kinetics studies reaction rates, in which chemical reaction aA+dD=gG+hH, the rate equation is $r=kc^a(A)c^d(D)$ [17]. It is the rate law, and may be expressed by (A,a)(d,D)=(G,r,H).

In transition state theory, A+BC=[A,B,C]→AB+C, in which [A,B,C] is transition state. It may be (A,k)(k',BC)=(A,B,C)=(AB,k'')C. It is also the repression of second oscillated state.

For catalyst, catalysis is A+C→M, M+B→C+P. It may be expressed by (A,k)(k',C)=M →M+B=(P,k)C, here C is catalysis. The acidic (A)-basic (B) catalysis is A+B=A:B. It is namely (A,k)(k',B)=(A,B).

In biological membrane, phosphoric acid has hydrophilicity

$$(R,k)(k',H_2O)=(R,j,H_2O), \tag{6}$$

and phosphatide has hydrophobicity (H,k)H_2O.

Van der Waals forces are the interactions between closed-shell molecules. They include the interactions between the partial charges of polar molecules and the repulsive interactions [17], and include induction force, dispersion force and orientation force (dipole-dipole force). The chemical bonds are lasting attractions between atoms, ions or molecules that enables the formation of chemical compounds, which may be ionic bonds, covalent bonds, metallic bonds as electromagnetic force, dipole-dipole interactions, the London dispersion force and hydrogen bonding.

In molecular interactions there is a widely used Mie potential:

$$V=\frac{C_n}{r^n}-\frac{C_m}{r^m}. \tag{7}$$

Here the first term represents repulsions and the second term attractions (for example, induction force). The corresponding force is:

$$F=-\frac{dV}{dr}=\frac{nC_n}{r^{n+1}}-\frac{mC_m}{r^{m+1}}. \tag{8}$$

A special example is the Lennard-Jones potential:

$$V = 4\varepsilon[(\frac{r_0}{r})^{12} - (\frac{r_0}{r})^6]. \tag{9}$$

The corresponding force is:

$$F = \frac{24\varepsilon}{r_0}[2(\frac{r_0}{r})^{13} - (\frac{r_0}{r})^7]. \tag{10}$$

For two molecules A and B, it may be repressed by the joint relation:

$$(A, \frac{C_0}{r^{m+1}})(-1, B) + (A, \frac{C_2}{r^{n+1}})B. \tag{11}$$

Different bonds exist in different crystal structures.

4. Applications in Biology

Mathematics in biology and medicine is an important developed direction, for example, the topological biology [18-20]. Fung, et al., investigated biomechanics, which includes the mechanical properties of bone, cartilage, ligament, tendon, muscle, the neuro-musculo-skeletal system, and biomaterials, etc [21-23]. But, biomechanics is not mechanics of rigid body. Since the same force acts at different points for biology, results are bigger difference, it may be expressed by F(p,B). Usual blood as an incompressible non-Newtonian fluid applies hydrodynamics [24-26], and is related with biorheology [27]. Bone corresponds to a classical linear elastic theory [28]. Biomembrances are researched from mechanics and thermodynamics [29]. Therefore, scalar and vector are applied widely in biology. Antonelli, et al., discussed the theory of sprays and Fisler spaces and their applications in physics and biology [30].

The theoretical basis of the traditional Chinese medicine (TCM) is Yin-Yang and Five-Element theory. Both mathematics are respectively the two-element replacement group and the five-element circular group SO(5) [31]. Further, we discussed space of Five Elements-Eight Trigrams in the traditional Chinese medicine, and proposed the pharmaceutic matrix mechanics [32].

Based on the inseparability and correlativity of the biological systems, we discussed the nonlinear whole biology and four basic hypotheses. It may unify reductionism and holism, structuralism and functionalism [33]. We proposed the extensive quantum biology [34], in which gene, cell, man and any live individual as the smallest live element in various levels are all different live quantum, and

discussed the loop quantum theory in biology and general biological string [33-35].

DNA is an important basis of molecular biology [36]. It is well-known that the model of DNA is a double helical structure. We assumed that A-T and G-C are the basic quantum of DNA, or assume nucleic acid has five types of base quantum: A, T, G, C and U, therefore, we proposed the extensive quantum theory of DNA [35]. For RNA the basic quantum elements are the corresponding A-U and G-C. The corresponding quantum theory and its many mathematical methods are applied to DNA and molecular biology. From this we discussed symmetry and supersymmetry of DNA, and the quantum theory and equations of DNA. Some solutions and functions of these theories may describe DNA, biological things and their motions [37]. Further, based on the extensive quantum theory of biology, the Schrödinger equation with the linear potential may become the Bessel equation. Its solutions are Bessel functions, and may form the double helical structure of DNA in three dimensional spaces. From this model we predicted the discrete bound energy spectrum of DNA. By using the entangled state representations of Bessel function [38] the investigations of formulation of DNA are easier, in which A-T and G-C are entangled each other. Moreover, we discussed some solutions of quantum mechanics and their meaning, and researched the entangled state of neurobiology by the extensive quantum method and the nonlinear theory [37].

We researched the neural synergetics, and based on the basic equations of synergetics, we derived quantitatively the Lorenz equations and Lorenz model of brain, whose two wings correspond to two hemispheres of brain, and two hemispheres jump about, which describes thinking. It shows that life lies in cooperation in chaos. The neural synergetics possesses three levels: structure, activity and thinking on brain. Based on the quantum and nonlinear neurobiology, we discussed the soliton-chaos double solutions of the nonlinear equation, and their possible meaning [39].

We discussed biofield and some nonlinear theories in biology, and researched chaos in biology and its application to cancer, and fractal and complex dimension in biology. We investigated nonlinear biothermodynamics and in which possible entropy decrease [40], and proposed the matrix representations of hypercycle theory, which can be defined by a degree of connectivity [41].

Bena, et al., searched loops in twistor space, and twistor transform in d dimensions and a unifying role for twistors [42]. We applied the loop quantum

theory to biology, and proposed the model of protein folding and lungs, and obtain four approximate conclusions [33]. Twistor is developed from general relativity, and may describe the curved space-time. We researched applications of twistor and its extensions in biology, which may describe some biological duality, and proposed specially the twistor model of DNA [43].

By using the simple joint relation, photosynthesis is:

$$(H_2O,\gamma)(\gamma,CO_2)=(CH_2O,O_2). \tag{12}$$

In biology enzymes act as catalysts. For example, carbonic anhydrase is:

$$(H_2O,e)(e,CO_2)=(H_2CO_3)e. \tag{13}$$

Generally, in the dynamics of proteins catalysis is the induced-fit and lock-and-key mechanism.

In cell biology cell division is $g(k,C)=(C_1,j)C_2$, or $=\begin{pmatrix}C_1\\C_2\end{pmatrix}$, here g is growth.

The protein conformations have four levels of structure (Fig.1): 1). Primary structure simply refers to the sequence of the different amino acids of the peptide or protein, which is held together by covalent or peptide bonds, which are made during the process of protein biosynthesis or translation. 2). Secondary structure is the general three-dimensional form of local segments of biopolymers such as proteins and nucleic acids (DNA/RNA) in molecular biology and structural biology. The secondary structures of peptides have one of two forms, α helix or β sheet. The α helix is a helical twisting of the polypeptide. α helix and β sheet involve interaction of the N-H of an amino acid with oxygen. It is namely the hydrogen bond. 3). Tertiary structure is the larger and more complicated conformation of the whole polypeptide. 4). Quaternary structure.

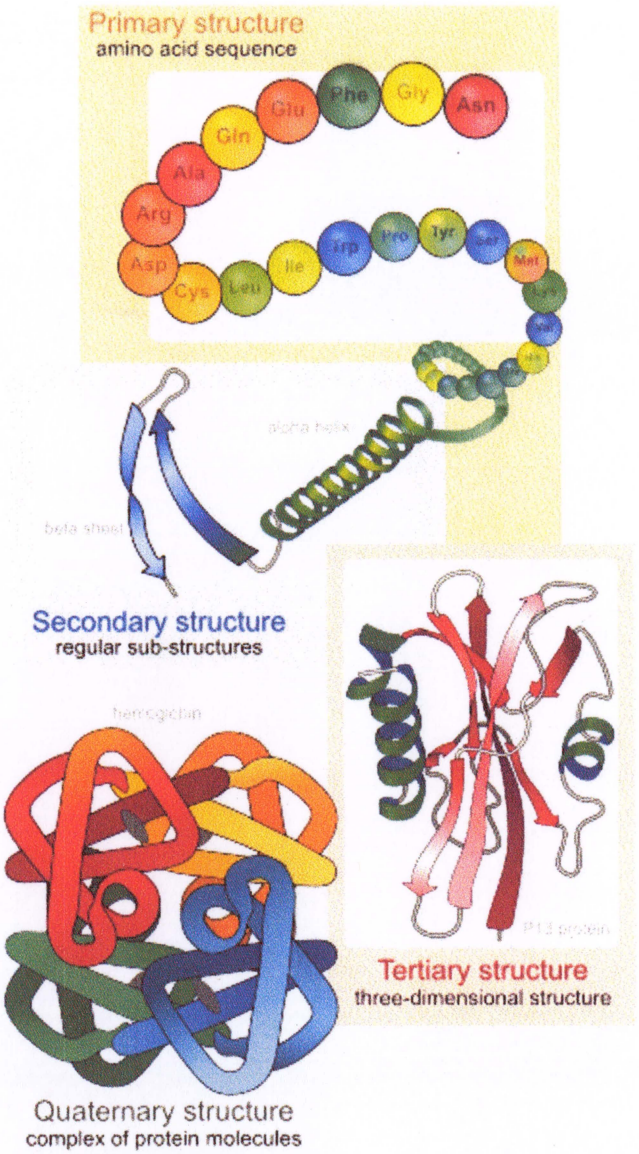

Fig.1. Protein structure from primary to quaternary structure.

Based on topological biology and structural biology, and combined the extensive quantum biology [34] and general biological string, we proposed that Calabi-Yau manifolds can provide a mathematical method to be applied to biology. Some Calabi-Yau spaces may possibly describe the biological spatial structures. In

biology usual Calabi-Yau manifolds are also smaller and cannot be observed except microscope. It is used to superstring and brane, so may also describe some biological strings, and biological branes, etc. Further, this may combine the extensive graph theory, which includes five types of the basic elements: various solid lines, dotted lines, wavy lines, and vertices, fields. Variegated Calabi-Yau manifolds and superstring-branes correspond to multiformity of biological structures [44].

In modern molecular biology [45-47] in order to make proteins, DNA is copied using another nucleic acid (RNA), which is transcription process. Its first step is gene expression, in which a particular segment of DNA is copied into RNA (especially mRNA) by the enzyme RNA polymerase. During transcription, a DNA sequence is read by an RNA polymerase, which produces a complementary, antiparallel RNA strand called a primary transcript.

Transcription proceeds in the following general steps: RNA polymerase binds to promoter DNA, and creates a transcription bubble, which separates the two strands of the DNA helix. This is done by breaking the hydrogen bonds between complementary DNA nucleotides. RNA polymerase adds RNA nucleotides, and RNA sugar-phosphate backbone forms with assistance from RNA polymerase to form an RNA strand. Hydrogen bonds of the RNA-DNA helix break, freeing the newly synthesized RNA strand. If the cell has a nucleus, the RNA may be further processed. This may include polyadenylation, capping and splicing. The RNA may remain in the nucleus or exit to the cytoplasm through the nuclear pore complex. When mRNA synthesis refers to from an RNA molecule, it is namely RNA replication, for instance, a viral RNA replicase [48].

Its mechanism is represented by figures of three different phases.

1) Initiation. Transcription begins with the binding of RNA polymerase (RNAP) as a core enzyme to the promoter in DNA.

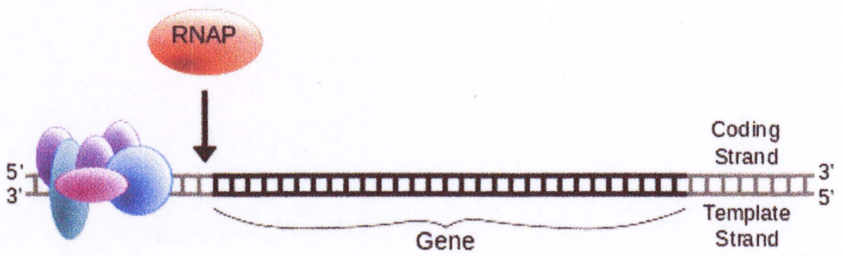

Fig.2. Simple diagram of transcription initiation.

2) Elongation.

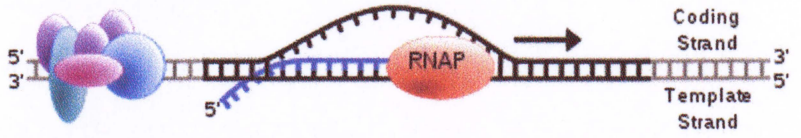

Fig.3. Simple diagram of transcription elongation (RNA in blue color)

3) Termination.

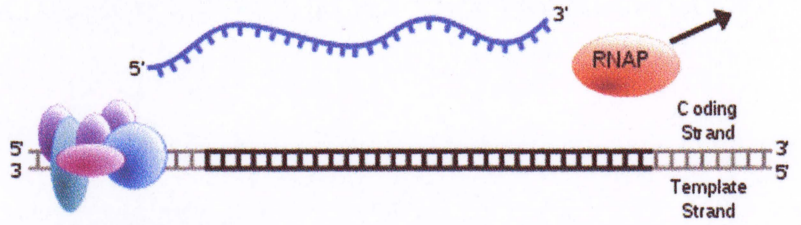

Fig.4. Simple diagram of transcription termination. RNA is shown in blue color.

We try to represent transcription by the simple form: Core enzyme (holoenzyme) joints the promoter e(p,D)=(D,R,D)e, here e is enzyme (RNAP), and p is promoter, then by (D,R,D)=(D,R)(R,D)=D'. It has two types: intrinsic (D,D')R and ρ (Rho protein)-dependent ρ (D,D')=(D,D')R.

Gene expressions under different conditions A and B are [46]:

$$\begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} A_1, A_2, A_3 \\ B_1, B_2, B_3 \end{pmatrix} = (C_1, C_2, C_3). \qquad (14)$$

The lactose metabolism is β (La)=(Ga,Gl). In regulation of gene expression the lac repressor binds to the operator in the regulatory upstream region of the operon. It is namely lac(o,G) and CAP(p,o)=(cAMP-CAP-complex,T).

Splicing is the process by which pre-mRNA is modified to remove certain stretches of non-coding sequences called introns; the stretches that remain include protein-coding sequences and are called exons. Sometimes pre-mRNA messages may be spliced in several different ways, allowing a single gene to encode multiple proteins. This process is called alternative splicing. Splicing is usually performed by an RNA-protein complex called the spliceosome, but some RNA molecules are also capable of catalyzing their own splicing (Fig.5). Splicing reaction of gene is:

$$e_1(i,e_2,G)=(e_1e_2,G)i, \tag{15}$$

here e is exon and i is intron.

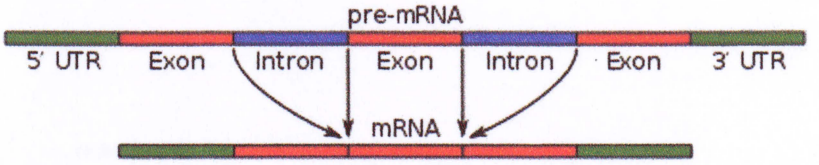

Fig.5. Simple illustration of a pre-mRNA, with introns (top). After the introns have been removed via splicing, the mature mRNA sequence is ready for translation (bottom).

RNA is used as a guide to put together the small molecules that make protein, which is translation process. This general progression (the central dogma) DNA→RNA→Protein is used in all life forms.

Mechanism of translation is: For prokaryotes, first the initiation is itR(sc, G)=(IF, G), here itR is initiator tRNA, sc is start codon, and IF is translation initiation factors. Then elongation and termination are

$$\text{itR(p,G)=(p')itR(A,G)=(p',A')itR(B,G)=...=(G')sc,} \tag{16}$$

here sc is stop codon.

The main types are: the helix-turn-helix structure, the zinc finger structure, the helix-loop-helix, (HLH) structure and the leucine zipper structure, etc. They are

$$\text{(R,k)(k',P)=(R, } j_i \text{,P), and } kk' = j_i \ (i=1,2,3,4). \tag{17}$$

Every cell in the human body contains exactly the same DNA. But, different cell types possess completely distinct functions and forms due to proteins determined by gene expression.

The genetic code may be an amino acid group with four U,C,A,G and three amino acid group with 64, which is a codon. Both correspond just to Four-Phases in Chinese Yi, and 64 Hexagrams (Guas), which is also codon of world. Therefore, Chinese traditional culture supposed that a body is a small universe, and Universe is a large body.

In DNA replication the semi-conservative replication is that each double-stranded DNA after replication conserves one parent strand but adds one new daughter strand. It is:

$$p(o,D_P)=\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}. \tag{18}$$

Different sensation systems are usually independent each other. Our collective open out the potential of blind children, and found through a period training of time, some children by touch or nose or ear can distinguish different colors, numbers and various figures. From this and other research, we proposed a hypothesis: The neural excitable cell is continuously induced and excited, then grow out new synapse and dendrite, and the feeling system, hearing system, smell system, etc., may joint to visual system, and form a new neural network, and achieve finally a transformation among vision and other sensations. It is expressed by nec(k,B)=(vn,B), here nec is neural excitable cell, and vn is new visual network. Further, we proposed some possible tests, for example, for trained mammal, etc., and researched possible theories [49,50]. It is a testable application of the nonlinear whole neurobiology. This may build a bridge between modern science and traditional culture, religion.

5. Applications in Social Science

In 1983 Weidlich and Haag published book *Concepts and Models of a Quantitative Sociology. The Dynamics of Interacting Populations* [51]. Then Lefebvre proposed the production of space [52]. Zohar and Marshall applied quantum physics, and proposed the quantum society [53]. Mol and Law [54] discussed regions, networks and fluids, in which there is wave-particle duality, and may apply to city, and develop to global wave and particle, and globally networks and fluids [55]. Various networks correspond to the social topology [56]. Globalization as the new structure with autopoiesis [57], auto-reproduced, self-making is a self-organization system. It should be a process with entropy decrease [58]. Further, we proposed quantitatively a universal entropy theory on evolution of any natural and social systems [59].

We discussed generally the four variables and the eight aspects in social physics, and searched social thermodynamics and the five fundamental laws of social complex systems [58]: the zeroth state fluctuation law; the first extensive energy E conservation law dE/dt=0 in an isolated system; the second extensive entropy S change law, S=klnW; the third threshold transition law; the fourth exclusion or inclusion law: Among various elements of any system the two different

relations exist, exclusion or inclusion, it may be expressed by (M,k)(k',N) or (M,k)N. And this system is stable when the former is larger the latter, and is unstable when the former is smaller the latter [60].

Usual social equations should be nonlinear, such we proposed the nonlinear whole sociology and its four basic laws [61], and discussed generally the social kinetic energy K, and the social potential energy U, and the social force F [62,61]. Based the social synergetics we proposed theorem of perfect correlation on humanity, and obtained the Lorenz model, which is a visualized two-party mechanism as a type of stable structure in democracy. A developed direction of society is the combination from macroscopic to microscopic order, from an actual capable handling to an ideal pursuance [63].

In micro-economics a main relation is expressed by $M(\frac{q}{p}, E_{ml})$, here M is market, q is quality and p is price. In macro-economics a main relation is expressed by $I(e, E_{ma})$, here I is income and e is employment. Governmental action is by G(p,E) here G is government and p is policy.

Using the similar formulas of the preference relation and the utility function, we proposed the confidence relations and the corresponding influence functions that represent various interacting strengths of different families, cliques and systems of organization. This produces a multiply connected topological economics [58]. It may be expressed by c(p,M)=(M,k)d, here c is confidence relation, p is power and d is a dark system separated. Further, we discussed the binary periods of the political economy by the complex function and the elliptic functions [58]. Society includes various states, but they cannot usually be gas states. Solid states possess bigger defects, and which diffuse easily, in particular, a pyramid structure. Modern society and persons are more analogy with liquid, therefore, we applied the social hydrodynamics [58]. Using the nonlinear equations of hydrodynamics we researched the formulations of the binary and multiple centers in various social systems [59]. It is expressed by:

$$(M,k)(k',N)=\begin{pmatrix}C_1\\C_2\end{pmatrix}\rightarrow\begin{bmatrix}C_1\\C_2\\\vdots\\C_n\end{bmatrix}. \tag{19}$$

Further, everyone possesses fate and luck. General relativity shows that matter and movement determine the space-time of everyone. This as a universal physical representation of causality is a great contribution to modern social science, therefore, we proposed the social extensive general relativity, which may describe some evolutional orbits of life and society [64].

Giddens proposed the duality of structure [65]. We proposed the social individual-wave duality, $(M,w)p \leftrightarrow (M,p)w$, and researched the social topology and the social strain field, and searched the pattern dynamics and the damage mechanics in sociology. From this mechanism the crisis of society may be described mathematically by the qualitative analysis method, and discussed some social phenomenon and war crisis [56]. Further, social hydrodynamics may develop to social magnetohydrodynamics, and social general relativity [64], etc.

Based on Chinese traditional philosophy and culture, a series of equations may correspond to Sky-Earth-Man, and we proposed the promotion-restraint sustainable developed pattern on the Five-Elements (Fig. 6), which include social (S) progress (democracy, justice, stabilization), economical development (F), science-technology (K), education-civilization (J) and environment-resource (H) [66,67].

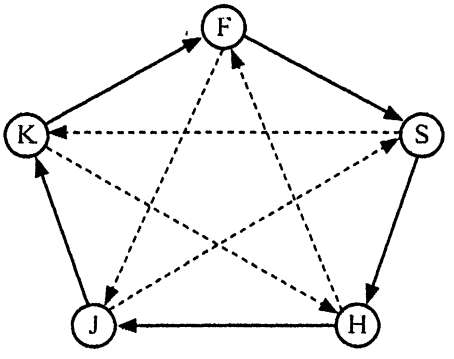

Fig.6. Promotion-Restraint Developed Pattern on the Five-Elements

Here a solid line represents a promotion relation M(k,N), and a dotted line

represents a restraint relation (M,k')N'. In graph theory the promotion and restraint relations are just a graph isomorphic to its complement. Various Five-Elements are fractals, and are isomorphic, etc., in algebraic topology.

Based on the basic social laws on energy and entropy, we discussed some social sustainable developed patterns: 1) The nonlinear limit and cycle pattern of three elements; 2) the synergetic pattern on society-economy-environment developed together; 3) the promotion-restraint pattern on Five-Elements. Their basis is lower entropy, even entropy decrease [66].

From this we should construct the stable social structure. We discussed various stable social structures of different levels. Chinese history shows a statistical rule, and forms a super-stable structure. The basis of the modern stable and harmonious society is equality. The comparison and combination at all times and in all over the world will redound to build a perfect society [60].

Various solutions of these equations may be: (1). Stable solutions, and correspond to stable society. (2). Periodic and quasi-periodic solutions, which may be periodic change or super-stability [60]. (3). Chaos solutions with microscopic not-rules and macroscopic rules. (4). Complete random.

Crises classify the city crises, the ecological crises, belief crises, etc. Crises diffuse to originate from some social potentials and forces. Social crises are mainly origin of inequality of fortune and income dispensation. Reduced difference is decrease potential. But, if there is not potential, social development will absent drive. Crises originate from mind, which is related with Buddhism [68] and psychology on happiness [69], etc.

General social joint relations are very complex, and possess prescription (for example, man-wife relation), stability and ordered (as grandfather, father, son, grandson), etc.

The husband-wife relation is usually one-to-one is the monogamy, and one-to-many is the plural marriage. General men-women relations have only repulsion of the same sex and attraction of the opposite sex, and the special attraction of the same sex, etc. These relations include stations, fortunes, features, even custom, lucky chance and so on. They may be matched for marriage, man with ability and beautiful woman, but, they may probably be opposite. In a word, kk' may be differ in thousands ways.

In social relations a soul-stirring change is shifty overturn sudden, for example,

Hitler and Stalin in Second World War, and Germany emperor William II (1859-1941) and Russian Grand Nicholas II (1868-1918) as brother-in-law, and Italy in First World War, whose happening may be expressed by (C,p)A, here C is Central, A is Allied, and p is Princip.

In a word, this simple mathematical method may be applied to many regions, and will be corrected and developed.

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